Volume optimization of a cuboid [36 marks]

***1.** [Maximum marks: 11]*

A piece of paper has 4 square edges of size x cut out. It is then folded up to make a cuboid with an open top.

(a) Find an expression for the volume of the cuboid formed when the original paper was a square of length 20cm.

[1]

(b) Use calculus to find the value of x that gives the maximum volume.

[3]

(c) By sketching a graph, find the value of x that gives the maximum volume when the original paper is a square with sides:

(i) 10cm.

(ii) 30cm.

[5]

(d) Hence find the value of x which gives a maximum volume of an m *by* m square.

[2]

***2.** [Maximum marks: 25]*

Next we will explore a 10 *by* n rectangle.

(a) Show that the value of x which gives the maximum volume is given by:

$$x = \frac{40 + 4n - 4\sqrt{(n-5)^2 + 75}}{24}$$

[7]

(b) Use this equation to find the value of x which gives the maximum volume for a 10 *by* 30 rectangle.

[2]

(c) We will now investigate what happens to this value of x as n tends to infinity, ie:

$$\lim_{n\to\infty} \frac{40 + 4n - 4\sqrt{(n-5)^2 + 75}}{24}$$

By making the substitution $n = \frac{\sqrt{75}}{u} + 5$ show that this can be written as:

$$\lim_{u\to 0} \frac{15 + \frac{\sqrt{75}}{u} - \sqrt{75}\frac{1}{u}\sqrt{1+u^2}}{6}$$

[4]

(d) Find the binomial expansion of $\sqrt{1+u^2}$ until the u^4 term

[3]

(e) Hence find the value that x approaches as n tends to infinity for an n *by* n rectangle

[5]

(f) By a similar process we can arrive at the following equation for an m *by* n rectangle

$$\lim_{u\to 0} \frac{\left(4m + 4\left(\frac{\sqrt{\frac{3m^2}{4}}}{u} + \frac{m}{2}\right)\right) - 4\sqrt{\frac{3m^2}{4}}\frac{1}{u}\sqrt{1+u^2}}{24}$$

Find the value that x approaches as n tends to infinity for an m *by* n rectangle

[4]