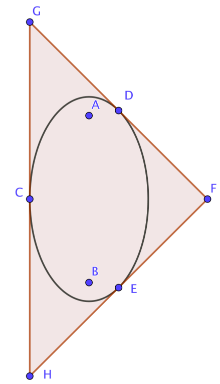
***Finding the Steiner inellipse using complex numbers [35 marks].***




***1.*** *[Maximum marks: 35]*

*The Steiner inellipse (pictured above) is the unique ellipse, which is tangent to the 3 midpoints of a triangle. We will explore a method of finding the equation of this ellipse using Marden's theorem.*

*(a)* *We start with a cubic equation:* $p(z) = z^3 - z^2 + z - 1$. *Write this in the form* $p(z) = (z - \alpha)(z^2 + \beta z + \gamma)$ *where* $\alpha \epsilon \mathbb{R}$*. Hence find the 3 roots of* $p(z)$.

*[7]*

*(b)* *Draw a sketch of these 3 roots on an Argand diagram. Join these 3 roots to make a triangle. Label this triangle with points F,G,H. Find the coordinates of the midpoints of the sides of this triangle. Label these as C, D, E on the sketch.*

*[3]*

*(c)* *Solve* $p'(z) = 0$. *Sketch these coordinates A, B on the Argand diagram.*

*[6]*

*We now treat our diagram as the real Cartesian x,y plane. Marden's theorem states that the* ***coordinates*** *A and B form the foci of the unique ellipse tangent to the triangle FGH at its midpoints C,D,E. This is illustrated at the top of the page. When A and B have coordinates* $(h, \pm c)$ *we have the following equation for our ellipse:*

$$\frac{(x-h)^2}{b^2} + \frac{y^2}{a^2} = 1$$

*Where:*

$$a^2 - b^2 = c^2.$$

*(d)* *Find a and b and hence find the equation for this ellipse.*

*[7]*

*(e)* *Show that the equation of the ellipse tangent to the midpoints of the triangle with vertices (0,2), (0,-2), (2,0) is given by:*

$$y^2 = \frac{4}{3} - 3\left(x - \frac{2}{3}\right)^2$$

*[8]*

*(f)* *Find the gradient of the ellipse at (1,1) and (1,-1). Comment on your answer.*

*[4]*