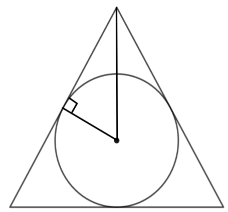
Approximating pi using circumscribed and inscribed circles. [39 marks]

The perimeter of a circumscribed polygon was used to find an approximation for pi by the ancient Greeks. They used the relationship $\pi \approx \frac{P}{2}$, where P is the perimeter of the regular polygon.

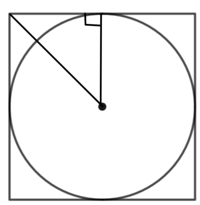


1. *[Maximum marks: 12]*

(a) *By considering the right-angled triangle drawn above, find the perimeter of equilateral triangle, whose sides are 3 tangents to the unit circle (radius 1). Hence use $\pi \approx \frac{P}{2}$ to find an approximation for π.*

[3]

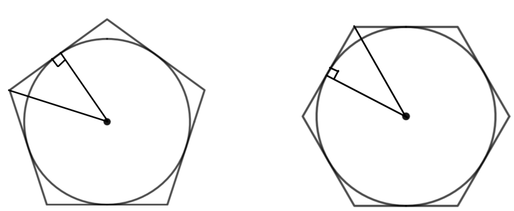
(b) *We can draw a square such that the perimeter of the square is made up of sides which are 4 tangents to the unit circle.*



By finding the perimeter of this square find a better approximation for π.

[2]

(c) *Use a circumscribed regular pentagon and regular hexagon to refine your approximations.*



[4]

(d) *Hence find the perimeter of an n-sided regular polygon circumscribed in a unit circle and the approximation for π for this polygon*

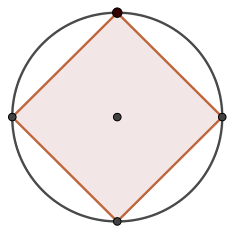
[3]

(e) *How many sides would a circumscribed polygon need to have to approximate π to 2 decimal places?*

[3]

2. *[Maximum marks: 11]*

We can use a similar method to approximate π when we have an inscribed regular polygon in a unit circle.



(a) *By considering an appropriate right-angled triangle, show that the perimeter of an n-sided regular polygon inscribed in a unit circle can be given by:*

$$P = 2n\sin\left(\frac{\pi}{n}\right)$$

[2]

(b) *How many sides would an inscribed polygon need to have to approximate π to 2 decimal places? Compare this with your answer to (v).*

[4]

(c) *Use L'Hopital's rule to find the limit as n approaches infinity for this approximation to π.*

[5]

The question continues on the next page

3. *[Maximum marks: 13]*

Archimedes was able to find upper and lower bounds for π by considering inscribed and circumscribed polygons without the need for trigonometry.

If we define i_n as the perimeter of an n-sided regular polygon inscribed in a circle and c_n as the perimeter of an n-sided regular polygon circumscribed in a circle then we have the following recursive relationship:

$$c_{2n} = \frac{2c_n i_n}{i_n + c_n}$$

$$i_{2n} = \sqrt{i_n c_{2n}}$$

(a) *By using your exact values for c_3 and i_3, and leaving your answers in an exact form, show a non-calculator method for finding*

(i) c_6 and i_6

[3]

(ii) c_{12} and i_{12}

[5]

(b) *The Indian astronomer Madhava discovered the following infinite series representation of π in the 1300s. This series is a convergent, alternating series.*

$$\pi = \sqrt{12}\sum_{k=0}^{\infty}\frac{(-3)^{-k}}{2k+1}$$

Any two consecutive terms of a convergent alternating series will give a lower and upper bound for the series.

Use this series to give both a lower and upper bound for π which are both correct to 2 decimal places. Compare your answer with (1v) and (2ii).

[5]