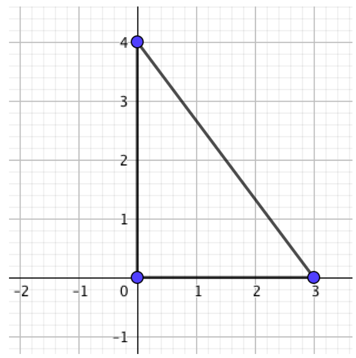
A sliding ladder investigation [40 marks]

1. *[Maximum marks: 16]*

In this question you investigate the motion of the midpoint of a sliding ladder. You have a 5 metre long ladder which ends rest against a perpendicular wall at (0,4) and (3,0).



(a) *Explain why the midpoint of the ladder is given by the coordinate (1.5, 2).* *[1]*

(b) *The ladder begins to slip down the wall such that the new base coordinate is (3+t, 0). Find the new height in terms of t.* *[1]*

(c) *Find the new midpoint (x,y) in terms of t. Write down an equation for the x coordinate in terms of t and an equation for the y coordinate in terms of t.* *[2]*

(d) *Eliminate t and show that an equation for the midpoint can be written as*

$$y^2 + x^2 = \frac{25}{4} \qquad x \geq 0, y \geq 0.$$

[3]

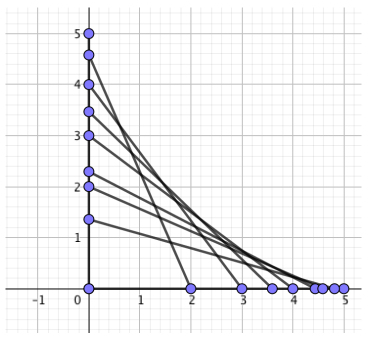
(e) *Sketch this equation and comment on its geometrical significance.* [3]

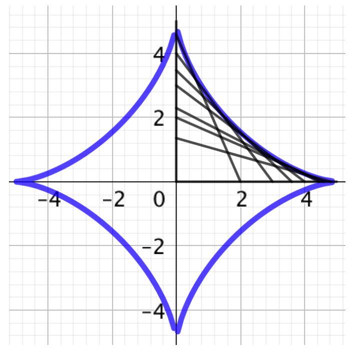
(f) *Find the general equation of the midpoint of a slipping ladder when the ladder rests against a perpendicular wall at (0,a) and (b,0).*

[6]

2. *[Maximum marks: 14]*

This time we take a ladder with length 5 and draw the family (envelope) of lines we get as it slips down the wall. The boundary of these lines creates the first quadrant of an astroid.





The equation of this particular astroid is given by:

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = 5^{\frac{2}{3}}$$

(a) *Find an equation for* $\frac{dy}{dx}$ *in terms of x only.*

[5]

(b) *Sketch* $\frac{dy}{dx}$ $x > 0$.

[3]

(c) *Find the gradient of the astroid when* $y = x$, $x > 0, y > 0$.

[4]

(d) *What is the significance of the line y = x in relation to the astroid in the first quadrant?*

[2]

This question continues on the next page.

3. *[Maximum marks: 10]*

We start with the astroid formed by a ladder of length m

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = m^{\frac{2}{3}}$$

We can define the curve parametrically as:

$$x = mcos^3\theta$$

$$y = msin^3\theta$$

with the length L of the astroid given by:

$$\frac{L}{4} = \int_0^{\frac{\pi}{2}} \sqrt{(\frac{dx}{d\theta})^2 + (\frac{dy}{d\theta})^2}\ d\theta$$

(a) *Find* $\frac{dx}{d\theta}$ *and* $\frac{dy}{d\theta}$

[2]

(b) *Hence show that:*

$$\frac{L}{4} = 3m\int_0^{\frac{\pi}{2}} sin\theta cos\theta\ \ d\theta$$

[4]

(c) *Find an equation for L in terms of m. Find the first quadrant length of the astroid made by a ladder of length 5.*

[4]