Mathematics SL Information for teachers and moderators

Copies of all Internal Assessment portfolio tasks published by the IB

Includes

First edition of previous TSM published 1998 First examinations May 2000
Current TSM published 2005 First examinations May 2006
Portfolio tasks for 2009/2010 published 2008

Note
Any tasks contained here CANNOT be submitted for final assessment after the November 2010 examination session

These tasks are NOT acceptable for assessment for examination sessions in 2011 and onward
1 Investigating the Quadratic Function

Work done on this assignment will be assessed against all six criteria A, B, C, D, E and F. You are therefore expected to use a graphic display calculator and/or computer for this assignment.

1 Sketch the graphs of
   (a) \( y = x^2 \)
   (b) \( y = x^2 + 3 \)
   (c) \( y = x^2 - 2 \).

What do you notice? Can you generalize?

2 Consider the graphs of
   (a) \( y = x^2 \)
   (b) \( y = (x - 2)^2 \)
   (c) \( y = (x + 3)^2 \).

What do you notice? Can you generalize?

3 Where would you expect the vertex on the graph of \( y = (x - 4)^2 + 5 \) to be? Explain why.

4 (a) Express \( x^2 - 10x + 25 \) in the form \( (x - h)^2 \).
   (b) Express \( x^2 - 10x + 32 \) in the form \( (x - h)^2 + g \).
   (c) Repeat this procedure with some examples of your own.
   (d) Describe a method of writing the quadratic expression \( x^2 + bx + c \) in the form \( (x - h)^2 + g \).

5 Describe the shape and position of the graph of \( y = (x - h)^2 + g \). Provide an explanation for this.

6 Do your findings apply to the graphs of other types of functions? Can you generalize?
2 Investigating the Graphs of Sine Functions

Work done on this assignment will be assessed against all six criteria A, B, C, D, E and F. You are therefore expected to use a graphic display calculator and/or computer for this assignment. Examples of graphs to support your findings should be included with your work.

Part 1

Look at the graph of \( y = \sin x \).

Compare the graphs of \( y = 2 \sin x \); \( y = \frac{1}{3} \sin x \); \( y = 5 \sin x \).

Investigate other graphs of the type \( y = A \sin x \).

How does the shape of the graph vary as \( A \) varies?

Express your conjecture in terms of

(a) transformation(s) of the standard curve \( y = \sin x \)

(b) characteristic(s) of the wave form.

Part 2

Investigate graphs of the type \( y = \sin Bx \) in a similar way.

Part 3

Investigate the family of curves \( y = \sin (x + C) \).

Part 4

Use your findings from parts 1, 2 and 3 to predict the shape and position of the graphs of

\( y = 3 \sin 2(x + 2) \); \( y = \frac{1}{2} \sin 3(x + 1) \); \( y = -\sin \frac{1}{2}(x - 1) \).

Check your predictions.

If \( y = A \sin B(x + C) \) explain how you can predict the shape and position of the graph for specific values of \( A \), \( B \), and \( C \).

Part 5

How is the graph of \( y = \cos x \) linked to the graph of \( y = \sin x \)?

What is the relationship between these functions?
3 Transforming Data

Work done on this assignment will be assessed against all six criteria A, B, C, D, E and F. You are therefore expected to use a graphic display calculator and/or computer for this assignment.

The purpose of this assignment is to investigate how translations and enlargements of data affect statistical parameters.

The table shows the heights in centimetres of 60 students.

<table>
<thead>
<tr>
<th>177</th>
<th>175</th>
<th>137</th>
<th>155</th>
<th>150</th>
<th>166</th>
</tr>
</thead>
<tbody>
<tr>
<td>132</td>
<td>146</td>
<td>179</td>
<td>140</td>
<td>169</td>
<td>177</td>
</tr>
<tr>
<td>141</td>
<td>148</td>
<td>130</td>
<td>176</td>
<td>135</td>
<td>130</td>
</tr>
<tr>
<td>157</td>
<td>172</td>
<td>178</td>
<td>143</td>
<td>143</td>
<td>136</td>
</tr>
<tr>
<td>132</td>
<td>166</td>
<td>130</td>
<td>151</td>
<td>145</td>
<td>178</td>
</tr>
<tr>
<td>131</td>
<td>171</td>
<td>160</td>
<td>140</td>
<td>179</td>
<td>166</td>
</tr>
<tr>
<td>145</td>
<td>142</td>
<td>177</td>
<td>176</td>
<td>132</td>
<td>135</td>
</tr>
<tr>
<td>164</td>
<td>179</td>
<td>161</td>
<td>145</td>
<td>134</td>
<td>179</td>
</tr>
<tr>
<td>139</td>
<td>149</td>
<td>135</td>
<td>142</td>
<td>172</td>
<td>148</td>
</tr>
<tr>
<td>159</td>
<td>160</td>
<td>137</td>
<td>130</td>
<td>130</td>
<td>164</td>
</tr>
</tbody>
</table>

1. Find the mean and standard deviation of the students’ heights.

2. Investigate how the mean and standard deviation change when
   (a) 5 cm is added to each height
   (b) 12 cm is subtracted from each height.

   How does adding $a$ to each score in any set of data change the mean and the standard deviation?

3. Investigate how the mean and standard deviation change when each height is multiplied by
   (a) 5
   (b) 0.2.

   How does multiplying each score by $a$ in any set of data change the mean and the standard deviation? What if $a < 0$?

4. Group the data into intervals and construct a cumulative frequency table from the grouped data. Draw a cumulative frequency curve and use it to find the median and interquartile range of the heights.
3 Transforming Data (continued)

5 Investigate how the median and interquartile range change when

(a) 5 cm is added to each height

(b) 12 cm is subtracted from each height.

How does adding \(a\) to each score in any set of data change the median and the interquartile range?

6 Investigate how the median and interquartile range change when each height is multiplied by

(a) 5

(b) 0.2.

How does multiplying each score by \(a\) in any set of data change the median and the interquartile range? What if \(a < 0\)?

7 Summarise your results and discuss their significance.

8 Use the results of your investigation to

(a) transform the given set of data so that it has a mean of 0

(b) transform the given set of data so that it has a standard deviation of 1

(c) transform the given set of data so that it has a mean of 0 and a standard deviation of 1.
4 An Investigation into the Newton-Raphson Method

Work done on this assignment will be assessed against criteria A, B, C, D and F. It is anticipated that a graphic display calculator will be used extensively in this assignment, and its use will be assessed.

1 Draw large neat diagrams to show how two iterations of the Newton-Raphson method produce successive approximations, \( x_1 \) and \( x_2 \), of the roots of the equations

(a) \( x^2 - 3 = 0 \), using a starting approximation \( x_0 = 2.2 \)

(b) \( x^2 - 3x^2 + 1 = 0 \), using a starting approximation \( x_0 = 1.5 \).

2 (a) Draw the graph of \( f(x) = 2 - \frac{1}{x^2} \), \( x \neq 0 \). Show graphically that if the starting approximation is \( x_0 = 2 \), the Newton-Raphson method will not produce convergent iterations, but if \( x_0 = 1 \) a set of convergent iterations is obtained.

(b) Find the number of iterations that will give the root of \( f(x) = 0 \) correct to five decimal places if \( x_0 = 1 \).

(c) Find the number of iterations that will give the root of \( f(x) = 0 \) correct to five decimal places if \( x_0 = 1.25 \).

3 Use the following functions

(a) \( g(x) = 3x - x^3 \)

(b) \( h(x) = x^2 - \cos x \)

(c) \( k(x) = e^{2x} - 3\cos x \)

and others of your choice, together with appropriate starting approximations, to investigate the conditions under which the Newton-Raphson method obtains a set of iterations which fail to converge to the required root. Comment on your results.
5 Transformation Matrices

Work done on this assignment will be assessed against the first five criteria A, B, C, D and E. While the use of a graphic display calculator may be helpful, the use of technology is not specifically required, nor will it be assessed.

1 Plot some points on a set of axes and join them to form a polygon. Check that your polygon has no symmetry. Write the position vectors of the vertices as a $2 \times n$ matrix. Call this matrix $X$.

2 Let $R$ be the matrix $egin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

(a) Calculate $X' = RX$.

(b) Plot the points corresponding to the columns of $X'$ and draw the resulting polygon.

(c) Describe the transformation which maps $X$ to $X'$.

3 Apply $R$ to other shapes. Comment on your results.

4 Investigate the transformations represented by other $2 \times 2$ matrices where each element is 0, 1, or $-1$. Try to find matrices for each of the origin-invariant transformations with which you are familiar.
6 Webs and Staircases: Investigating Fixed-point Iteration

Work done on this assignment will be assessed against all six criteria A, B, C, D, E and F. You are therefore expected to use a graphic display calculator and/or computer for this assignment.

Note: This assignment assumes that you have previously constructed simple staircase and web diagrams using \( x = f(x) \), where \( f(x) \) is a linear function.

1. The graph below can be used to illustrate a fixed-point iteration for the iterative sequence \( x_n = 0.5x_{n-1} + 5 \).

![Graph](image)

\[ f(y) = 0.5x + 5 \]

\[ f(y) = x \]

(a) Copy the graph and, starting at \( x_0 = 1 \), draw the staircase corresponding to the fixed-point iteration.

(b) Copy and complete the table below for the subsequent values of \( x_n \).

<table>
<thead>
<tr>
<th>( n )</th>
<th>( x_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td></td>
</tr>
</tbody>
</table>

6 Webs and Staircases: Investigating Fixed-point Iteration (continued)

(c) The iteration is approaching a limit, which is the solution of an equation. What is the equation? Verify that the limit is the solution of this equation.

2 (a) Consider the iteration $x_n = Kx_{n-1} + 5$. By varying the value of $K$ from $-2$ to $2$, $K \neq 0$, investigate the relation between $K$ and the convergence of the iteration.

Use your graphic display calculator to assist you in your investigation but include hand-drawn graphs for $K = -2, -1, -0.5$ and $2$.

Describe the results of the investigation. What do you think will happen if $K = 1$?

(b) Form a conjecture about the slope of the generating function and the convergence (or divergence) of the iteration. Here, the generating function is $f(x) = Kx + 5$.

Provide other examples to support your conjecture.

3 (a) Repeat the investigation using the iteration $x_n = Kx_{n-1}(1-x_{n-1})$ for $0 < K < 4$.

Describe the effect $K$ has on the convergence. Illustrate different results with hand-drawn graphs. Choose different values for $x_0$ and discuss how the choice of $x_0$ affects the convergence.

(b) Solve the equation $Kx(1-x) = x$ in terms of $K$.

Now find the derivative of $f(x) = Kx(1-x)$ in terms of $K$ and determine an expression for the derivative at the fixed point.

Explain how these results relate to the results obtained in your investigation of $x_n = Kx_{n-1}(1-x_{n-1})$. 
7 Radio Transmitters

Work done on this assignment will be assessed against the first four criteria A, B, C and D. While the use of a graphic display calculator may be helpful, the use of technology is not specifically required, nor will it be assessed.

(November 1995, Paper 2)

1 A radio transmitter sends signals to a railway engine which runs along a straight track. When a set of coordinate axes is used to represent this system, the transmitter is at R(1, 0) and the track T is a line with equation $2x + y = 30$, where the units are kilometres.

(a) Draw a set of coordinate axes, using the same scale for both axes. Plot R and draw the graph of T. Clearly label with their coordinates the points A and B at which T crosses the x-axis and the y-axis, respectively.

The engine receives the strongest signal at C, which is the point on T closest to R.

Let P(x, y) represent a general point on T.

(b) (i) Express $y$ in terms of $x$.

(ii) Find, in terms of $x$, the vector $\vec{RP}$.

(iii) Use the scalar product $\vec{RP} \cdot \vec{BA}$ to find the coordinates of C.

(c) As an alternative to the vector method of part (b), let $s(x)$ represent the distance between R(1, 0) and P(x, y).

(i) Show that $s(x) = \sqrt{5x^2 - 122x + 901}$.

(ii) Find $s'(x)$.

(iii) Solve the equation $s'(x) = 0$. Explain why this gives the $x$-coordinate of C.

When the engine is more than 28 km away from the radio transmitter, the signal is too weak to be effective.

(d) (i) Draw an arc with centre R and radius 28 to show the points on T which are within the effective range of the transmitter.

(ii) Find the length of the portion of T along which the engine will be within the effective range of the transmitter.
7 Radio Transmitters (continued)

(May 1996, Paper 2)

2 The diagram represents the relative positions of the centres of three towns Agosham (A), Buckersfield (B) and Chetterham (C) which are linked by straight roads. Buckersfield is 26 km from Agosham and 10 km further north, and Chetterham is 42 km due south of Buckersfield.

Diagram not to scale

(a) How far east of Agosham is Buckersfield?

(b) Find the exact value of \( \cos(\hat{A}\hat{B}\hat{C}) \). Hence, or otherwise, find the distance of Chetterham from Agosham.

Let \( \vec{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) represent a displacement of 1 km due east and \( \vec{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \) represent a displacement of 1 km due north.

(c) Write down the vectors

(i) \( \vec{AB} \)

(ii) \( \vec{AC} \).

(d) The range of 'Radio Buckersfield', broadcast from B, is such that it can just be heard at only one point P on the Agosham-Chetterham road, \( d \) km from A. Write down an expression in terms of \( d \) for

(i) the vector \( \vec{AP} \)

(ii) the vector \( \vec{PB} \).
7 Radio Transmitters (continued)

(e) Find an expression in terms of $d$ for the scalar product $\vec{AP} \cdot \vec{PB}$, and hence or otherwise

(i) find the value of $d$

(ii) find the range of ‘Radio Buckersfield’.

(f) A new transmitter is to be set up at E, equidistant from all three towns.

(i) Explain why $\overrightarrow{BEC} = 2\overrightarrow{BAC}$.

(ii) Use part (f)(i) to find how far from each town the transmitter must be.

In each of these questions a simple model for radio transmissions is assumed. Comment on the possible limitations of this.
The Decibel Scale

Work done on this assignment will be assessed against the first four criteria A, B, C and D. While the use of a graphic display calculator may be helpful, the use of technology is not specifically required, nor will it be assessed.

The intensity $I$ of a sound wave is measured in watts per metre squared ($\text{W m}^{-2}$). The lowest intensity that the average human ear can detect, i.e. the threshold of hearing, is denoted by $I_0$, where $I_0 = 1 \times 10^{-12}$ $\text{W m}^{-2}$. The loudness of sound, i.e. its intensity level $\beta$, is measured in decibels (dB), where

$$\beta = 10 \log_{10} \frac{I}{I_0}.$$ 

1. Find the intensity of ordinary conversation which has an intensity level of 65 dB.

2. The sound inside an automobile travelling at 90 km h$^{-1}$ has an intensity level of 75 dB. Find the intensity of this sound source in W m$^{-2}$.

3. Copy and complete the table below.

<table>
<thead>
<tr>
<th>Source of Sound</th>
<th>Intensity Level (dB)</th>
<th>Intensity (W m$^{-2}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jet plane at 30m</td>
<td>140</td>
<td></td>
</tr>
<tr>
<td>Threshold of pain</td>
<td></td>
<td>1</td>
</tr>
<tr>
<td>Loud indoor rock concert</td>
<td>120</td>
<td>$1 \times 10^{-2}$</td>
</tr>
<tr>
<td>Siren at 30m</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Busy street traffic</td>
<td>70</td>
<td>$1 \times 10^{-8}$</td>
</tr>
<tr>
<td>Quiet radio</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Whisper</td>
<td>20</td>
<td></td>
</tr>
</tbody>
</table>

4. Describe the relationship between an increase in intensity and the corresponding increase in intensity level.

5. Using the given formula, $\beta = 10 \log_{10} \frac{I}{I_0}$, show that this relationship holds true for any increase in intensity.

6. (a) Sketch the graph of $\beta$ as a function of $I$.

   (b) Determine the intensity of a sound wave whose intensity level is

   (i) 6 dB
   (ii) 12 dB
   (iii) 18 dB.
(c) How many times more intense would the 18 dB sound seem compared to the 6 dB sound?

7 A dog's threshold of hearing is $1 \times 10^{-13}$ W m$^{-2}$.

Discuss the following points.

(a) How does the graph change? Why does it change?

(b) How does the relationship described in part 4 change? Why does it change?

(c) When you and your dog cross a busy street does the noise seem equally loud to both of you? Explain your answer.
9 Equations of Lines in Vector Form

Work done on this assignment will be assessed against the first four criteria A, B, C and D. While the use of a graphic display calculator may be helpful, the use of technology is not specifically required, nor will it be assessed.

Consider the vector \( \mathbf{r} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ -3 \end{pmatrix} + t \begin{pmatrix} 1 \\ 3 \end{pmatrix} \) where \( t \in \mathbb{R} \).

1. Let \( t \) take various values and draw the resulting vectors on a cartesian coordinate system. What do you notice?

If \( R(x, y) \) is the point whose position vector is \( \mathbf{r} \), how could you describe the set of all possible points \( R \)?

2. Repeat this investigation with \( \mathbf{r} = \begin{pmatrix} -2 \\ 6 \end{pmatrix} + t \begin{pmatrix} 4 \\ -3 \end{pmatrix}, \ t \in \mathbb{R} \). Compare the results to those obtained in part 1.

3. Choose any two points in the plane and draw the line \( L \) that passes through them.

   (a) Express \( L \) as \( \mathbf{r} = \mathbf{p} + t \mathbf{d} \) where \( \mathbf{p} \) and \( \mathbf{d} \) are to be determined.

   (b) Is this a unique representation of \( L \)? If not, give other examples, in vector form, of equations of \( L \), verifying that these are indeed the same line.

   (c) Describe in words how a line can be expressed in vector form.

4. A line \( L \) passes through a point \( U(h, k) \) in a direction \( \mathbf{d} = \begin{pmatrix} a \\ b \end{pmatrix} \).

   (a) Write down \( \mathbf{r} = \begin{pmatrix} x \\ y \end{pmatrix} \), a general position vector for point \( R(x, y) \) on \( L \).

   (b) Express \( \mathbf{r} \) in terms of \( h, k, a, b \) and \( t \).

   (c) By solving for \( t \) show that the equation for \( L \) can be written in the form \( cx + dy + e = 0 \), \( c, d, e \in \mathbb{R} \).

5. A line \( L \) passes through a point \( U(h, k) \) in a direction \( \mathbf{d} = \begin{pmatrix} a \\ b \end{pmatrix} \). Using a suitable vector \( \mathbf{n} \), which is normal (perpendicular) to \( L \), show that \( L \) can be expressed in the form \( (\mathbf{r} - \mathbf{u}) \cdot \mathbf{n} = 0 \), where \( \mathbf{r} = \begin{pmatrix} x \\ y \end{pmatrix} \). Explain clearly, using diagrams to support your explanation.
10 Modelling a Can of Drink

Work done on this assignment will be assessed against the first four criteria A, B, C and D. While the use of a graphic display calculator may be helpful, the use of technology is not specifically required, nor will it be assessed.

In this assignment you will be trying to find the dimensions of a regular drink can which minimizes the amount of metal used to make the can, and comparing it with the actual dimensions.

1  (a) Consider a drink can and note its capacity from the information given on the can (e.g. 355 ml).

   (b) Describe carefully how your can differs from a perfect cylinder made of metal of constant thickness.

2  (a) Assuming your can is a perfect cylinder of constant thickness, use the value of the capacity to write a relation between the radius, \( r \) cm, and the height, \( h \) cm.

   (b) Write down the total surface area, \( S \) cm\(^2\), of the can in terms of \( r \) and \( h \).

   (c) Use parts 2(a) and 2(b) to find \( S \) as a function of \( r \) only.

3  (a) Differentiate \( S \) with respect to \( r \) and solve \( S'(r) = 0 \). Find the corresponding value of \( h \). Show that this value of \( r \) gives a minimum rather than a maximum value of \( S \).

   (b) Use another method to check your answers to part 3(a).

4  Compare the values of \( r \) and \( h \) found in part 3 with the actual dimensions of your can. Comment on your findings.

5  (a) Now assume that the ends of your can are twice as thick as the curved side. If the volume of the metal used to form the can is \( V \) cm\(^3\), write down \( V \) as a function of \( r \), assuming that the thickness of the curved portion of the can is \( t \) cm (or if you prefer take an actual value for \( t \) of 0.001 cm).

   (b) Repeat parts 3 and 4 for \( V \).

   (c) Compare your new values of \( r \) and \( h \) with the actual dimensions of your can and comment on the suitability of the revised model as a representation of the can.
11 Population Growth

Work done on this assignment will be assessed against all six criteria A, B, C, D, E and F. You are therefore expected to use a graphic display calculator and/or computer for this assignment.

1 A sample of 100 bacteria are multiplying so that their population doubles in size every three minutes.

(a) Complete the table below for $t =$ time in minutes and $A =$ number of bacteria.

<table>
<thead>
<tr>
<th>$t$</th>
<th>$A$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>100</td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td></td>
</tr>
<tr>
<td>15</td>
<td></td>
</tr>
</tbody>
</table>

(b) Plot the ordered pairs $(t, A)$ on a cartesian coordinate system.

2 (a) Express $A$ as a function of $t$ in the form $A = A_0b^t$ where $A_0 =$ initial population and $b$ and $c$ are to be determined.

(b) Rewrite $A$ as a function of $t$ in the form $A = A_0e^{kt}$ where $k$ is to be determined.

(c) Graph the function in (b) on a graphic display calculator and compare it to the graph plotted in 1(b).

Use the features of your graphic display calculator to determine
(i) the values of $A$ when $t = 5$, $t = 20$, and $t = 100$

(ii) the values of $t$ when $A = 1000$, $A = 10000$, and $A = 1000000$.

Discuss any problems you encountered and the reasonableness of your results in (i) and (ii).

How well does the graph of $A$ represent the actual growth pattern of the bacteria population? What might occur in reality?

(iii) Use the results of (ii) to determine when the population is first greater than

(a) 1000

(b) 10000

(c) 1000000.
3 (a) Assume that, at each stage of growth, 20% of the existing population dies before
doubling takes place. Complete the table below under these conditions, rounding
each answer to the nearest whole number.

<table>
<thead>
<tr>
<th>t</th>
<th>A</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>100</td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td></td>
</tr>
<tr>
<td>15</td>
<td></td>
</tr>
</tbody>
</table>

(b) Modify the function from 2(b) to reflect this new situation. Graph this function on
your graphing calculator along with the original version from 2(c).

(c) Sketch both functions on the same set of axes, indicating an appropriate domain and
range.

(d) Describe the changes in the size of the bacteria population under each condition as
time progresses.

4 Population growth can be more realistically modelled using a function of the form

\[ P = \frac{k}{1 + ae^{-\lambda t}} \]

where \( k, a, \) and \( \lambda \) are constants, and \( t \) is the time.

(a) Consider the function \( P = \frac{1000000}{1 + 99999e^{-0.4t}} \).

(i) Find the value of \( P \) when \( t = 0 \).

(ii) What value does \( P \) tend towards as \( t \) grows very large?

(iii) Use the information from (i) and (ii) to determine an appropriate window
and draw the graph of \( P \) using a calculator.

(iv) Determine the values of \( P \) when \( t = 20, 50 \) and \( 100 \).

(b) Explain how such a function better represents population growth under actual
conditions.

(c) Investigate the graphs of other functions \( P = \frac{1000000}{1 + 99999e^{-\lambda t}} \) by taking different values
of \( \lambda \). What effect does the choice of \( \lambda \) have on the graph of \( P \)?
11 Population Growth (continued)

(d) Repeat the investigation in (c) by setting $\lambda = 0.4$ and varying the value of $a$. What effect does this have?

(e) Repeat the investigation in (c) by setting $\lambda = 0.4$ and $a = 9999$ and varying the value of $k$. What effect does this have?

(f) Determine a combination of $k$, $a$ and $\lambda$ so that $P = 1000$ when $t = 0$ and the limiting value is 500 000.

5 Comment on the advantages and disadvantages of the different models.
12 Investing Money

Work done on this assignment will be assessed against the first four criteria A, B, C and D. While the use of a graphic display calculator may be helpful, the use of technology is not specifically required, nor will it be assessed.

1. Alan invests $1000 at an interest rate of 12% per year. Copy and complete Table 1 which shows $A$, the value of the investment in dollars after $t$ years, assuming that the interest is compounded yearly.

<table>
<thead>
<tr>
<th>$t$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>5</th>
<th>10</th>
<th>20</th>
<th>30</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>1000</td>
<td>1254.40</td>
<td>3105.85</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2. (a) Show that if interest of 12% per year is compounded monthly it is equivalent to an interest rate of approximately 12.68% per year if the interest is compounded yearly.

(b) Copy and complete Table 2 which shows $B$, the value of the investment in dollars after $t$ years, assuming that the interest is compounded monthly.

<table>
<thead>
<tr>
<th>$t$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>5</th>
<th>10</th>
<th>20</th>
<th>30</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B$</td>
<td>1000</td>
<td>1269.68</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

3. Draw, on the same axes, graphs of the results from Table 1 and Table 2.

4. Investigate the effect on the value of the investment if interest is compounded more frequently.

5. Barbara invests $200 at the start of each year for a period of 5 years, at an interest rate of 12% per year. Thereafter, she stops adding $200 at the start of each year but leaves her investment in the bank to earn interest at 12% per year.

Table 3 shows $C$, the value of the investment in dollars after $t$ years, assuming that the interest is compounded yearly.

<table>
<thead>
<tr>
<th>$t$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>5</th>
<th>10</th>
<th>20</th>
<th>30</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C$</td>
<td>200.00</td>
<td>424.00</td>
<td>674.88</td>
<td>1423.04</td>
<td>2507.88</td>
<td>7789.09</td>
<td>24191.72</td>
</tr>
</tbody>
</table>

Draw a new table showing the value $D$, Barbara’s investment in dollars after $t$ years, assuming that the interest is compounded monthly. Label it Table 4.
If Barbara starts her investment at the beginning of 1990 so that her final premium of $200 is paid at the beginning of 1994 and Alan’s single investment is made at the beginning of 1994, draw on the same axes graphs of the values of Alan’s and Barbara’s investments during the interval of years from 2000 to 2010. You may assume that interest is compounded yearly.

Explain why Barbara’s investment is worth more than Alan’s.
13 The Water in a Lake

Work done on this assignment will be assessed against all six criteria A, B, C, D, E and F. You are therefore expected to use a graphic display calculator and/or computer for this assignment.

A lake has a capacity of 60000 m³ of water. At the beginning of a particular year there are 50000 m³ of water in the lake.

A stream enters at one end with a flow rate of \( F = 300 \left( 1 + \cos \frac{2\pi t}{365} \right) \) m³ per day, where \( t \geq 0 \) represents the day of the year, with \( t = 0 \) on 1 January. Another stream at the other end empties the lake.

Additional water is added to the lake through rainfall according to the formula \( R = 600 \left( 1 + \cos \frac{2\pi t}{365} \right) \) m³ per day.

Water evaporates from the lake according to the formula \( E = 500 \left( 1 + \sin \frac{2\pi t}{365} \right) \) m³ per day.

Suppose that water exits the lake from the second stream at a rate of 300K m³ per day.

1. (a) Write down a function, \( N(t) \), which represents the net daily flow of water into the lake.

(b) Investigate graphs of this function for various values of \( K \). Include graphs for values of \( K \) of 0.5, 1, and 2.

(c) Use your graph for \( K = 1 \) to find the total volume of water in the lake on 30 June for this particular model.

(d) Write down a function \( V(t) \) which represents the total volume of water in the lake on day \( t \).

(e) Investigate graphs of this function for various values of \( K \). Include graphs for values of \( K \) of 0.5, 1, and 2.

(f) A nearby town will be flooded if the volume of water exceeds 70000 m³ while the lake will suffer irreversible ecological damage if the volume falls below 45000 m³. Determine the values of \( K \) that will result in these levels.

2. Suppose now that the flow of water exiting the lake can be controlled throughout the year according to the formula \( C = 300 \left( 1 + \frac{Kt}{365} \right) \) m³ per day.

(a) Rewrite \( V(t) \) with this change and sketch its graph for values of \( K \) of \(-1\), \(-0.5\), \(0.5\), and \(1\). What do you notice?
(b) Discuss the effect this type of control would have on the flood problem and the low-water problem. What value(s) of \( K \) would prevent these problems?

3 Now suppose that the exit flow can be controlled according to the formula

\[
X = 300 \left( 1 + K \sin \frac{2\pi t}{365} \right) \text{ m}^3 \text{ per day}.
\]

(a) Investigate the revised \( V(t) \) for different values of \( K \). What do you notice?

(b) Sketch graphs of \( V(t) \) for values of \( K \) of \(-1, 1, \) and \(2\). What value(s) of \( K \) will prevent the high and low-water problems mentioned above?

4 Investigate whether these control measures will be effective for differing amounts of water in the lake at the start of the year.
Bill was returning home from a conference when he received a speeding ticket for travelling at 123 km h\(^{-1}\), according to the police radar, in a zone where the speed limit was 90 km h\(^{-1}\). The diagram below shows the location of the radar on a 3 km section of highway, measured from the front of Bill’s car which is at the traffic lights.

Bill felt that the police had made a mistake for two reasons. First of all, Bill is a very careful driver, and in fact, this was his first speeding ticket in over 20 years of driving, and he therefore considered the 33 km h\(^{-1}\) over the limit to be quite excessive. Secondly, all the cars passing through this zone were measured by the radar and all violators, 5 km h\(^{-1}\) or more over the limit, were stopped and received speeding fines. Bill was following Art, the drama teacher, the whole way home from the conference and, surprisingly, Art was not stopped for violating the speed limit. Upon returning home (after paying his fine, of course), Bill decided to analyse the situation more carefully.

For this problem, you need to complete the following analysis.

1. Convert the speed limit (90 km h\(^{-1}\)) and Bill’s speed (123 km h\(^{-1}\)) into metres per second (m s\(^{-1}\)).

2. Suppose that both Bill and Art proceeded when the traffic lights turned green, that is when \(t = 0\). Bill started approximately 10 m behind Art and suppose that Bill’s distance \(d\), measured in metres from the traffic lights, at any time \(t\), measured in seconds, is given by the following function:

\[
d(t) = \begin{cases} 
0.85t^2, & 0 \leq t < 25 \\
26t - 119, & t \geq 25 
\end{cases}
\]
14 Speed Limits (continued)

(a) Show that Bill passes the police radar when \( t = 20 \).

(b) Show that if Bill was not stopped by the police, he would have travelled approximately 3 km when \( t = 120 \).

(c) Bill’s average velocity can be calculated using \( v = \frac{\Delta d}{\Delta t} \). Determine his average velocity from \( t = 0 \) to \( t = 20 \) and from \( t = 0 \) to \( t = 120 \).

(d) Determine Bill’s average velocity between the following times, by copying and completing the following chart:

<table>
<thead>
<tr>
<th>Time (s)</th>
<th>( \Delta d )</th>
<th>( \Delta t )</th>
<th>Average velocity (ms(^{-1} ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t = 0 ) to ( t = 20 )</td>
<td>340</td>
<td>20</td>
<td>17</td>
</tr>
<tr>
<td>( t = 1 ) to ( t = 20 )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( t = 5 ) to ( t = 20 )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( t = 10 ) to ( t = 20 )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( t = 15 ) to ( t = 20 )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( t = 19 ) to ( t = 20 )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( t = 19.5 ) to ( t = 20 )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( t = 19.9 ) to ( t = 20 )</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The radar measures a car’s velocity over a very short period of time. We can measure Bill’s velocity at exactly 20 seconds, by calculating the following limit for \( d(t) \), when \( 0 \leq t < 25 \), at \( t = 20 \):

\[
\lim_{h \to 0} \frac{d(t+h) - d(t)}{(t+h) - t}
\]

3 Using your results from part 2, carefully analyse what has happened. Should Bill have been issued with a speeding ticket?

4 Using a suitable scale, accurately sketch the graph of \( d(t) \), for \( 0 \leq t < 30 \). Put \( t \) on the horizontal axis.

5 Create a continuous distance function \( A(t) \), for Art, the drama teacher, which satisfies the following three conditions given in this problem.

- \( A(0) = 10 \) (remember Art started 10 m in front of Bill).
- Art does not violate the speed limit at the point when he passes the police radar.
- Bill never passes Art, for the entire 3 km stretch of highway.

6 Sketch the graph of \( A(t) \) on the same set of axes as \( d(t) \). Demonstrate that the conditions on the function \( A(t) \) (from part 5) have been satisfied.
15 Crossing a River

Work done on this assignment will be assessed against the first four criteria A, B, C and D. While the use of a graphic display calculator may be helpful, the use of technology is not specifically required, nor will it be assessed.

The diagram represents a river 1 km wide.

On one bank, at the origin of the axes O, is a boat which travels at a velocity of $v_W = \begin{pmatrix} u \\ v \end{pmatrix}$ relative to the water. The point A is directly opposite the origin on the opposite bank of the river. The current, $c$, flows in the positive $x$ direction. The velocity of the boat relative to the bank is $v_B = \begin{pmatrix} u_B \\ v_B \end{pmatrix}$.

1. (a) Find $v_B$ when $v_W = \begin{pmatrix} 0 \\ 5 \end{pmatrix}$ and $c = \begin{pmatrix} 12 \\ 0 \end{pmatrix}$, and draw a triangle to represent these vectors.

(b) Find the time, $t$, for the boat to cross the river when $v_W$ and $c$ have the values given above. Recall that $t = \frac{\text{distance}}{\text{speed}}$.

(c) Find the distance, $s$, downstream of A where the boat reaches the opposite bank.

2. If $u = 1$, and $|v_W| = 5$, find $v$, and find a relationship between $u$ and $v$.

3. Copy and complete the following table for $c = \begin{pmatrix} 12 \\ 0 \end{pmatrix}$ and $|v_W| = 5$.

<table>
<thead>
<tr>
<th>$u$</th>
<th>$v$</th>
<th>$u_B$</th>
<th>$v_B$</th>
<th>$t$</th>
<th>$s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>5</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>$\sqrt{21}$</td>
<td>14</td>
<td>4.582</td>
<td>0.218</td>
<td>3.055</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\sqrt{21}$</td>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
15 Crossing a River (continued)

4 If $t(u)$ is the time taken for the boat to reach the opposite bank find $t$ as a function of $u$.

5 (a) If $s$ is the distance from A to the point where the boat reaches the opposite bank, show that $s(u) = \frac{u + 12}{\sqrt{25 - u^2}}$ for $-5 < u < 5$.

(b) Draw the graph of $s(u)$ and explain in words why it is the shape it is.

(c) Find the value of $u$ for which $s$ is a minimum.

6 (a) Write down the scalar product $v_w \cdot v_B$ in terms of $u$ and find the value of $u$ for which $v_w \cdot v_B = 0$. Comment on your result.

(b) Draw a diagram of $v_w$, $v_B$ and $c$ when $v_w \cdot v_B = 0$. 
2 Absolute value (modulus) graphs (continued)

3 (a) If you were given a graph of $y = f(x)$, describe how you would draw the graphs of $y = |f(x)|$ and $y = f(|x|)$. Illustrate your answer by copying and completing the following graph to show all three curves, $y = f(x)$, $y = |f(x)|$ and $y = f(|x|)$.
2 Absolute value (modulus) graphs

Work done on this assignment will be assessed against criteria A, B, C, D and F. You are therefore expected to use a graphic display calculator and/or computer for this assignment.

1 (a) Compare the graphs of the following pairs of functions, by sketching each pair on the same axes.
   (i) \( y = x + 1 \) and \( y = |x + 1| \)
   (ii) \( y = x^2 - 3x + 2 \) and \( y = |x^2 - 3x + 2| \)
   (iii) \( y = \sin x - 0.5 \) and \( y = |\sin x - 0.5| \)

(b) In each case, when the graph of \( y = |f(x)| \) is drawn, describe what happens to those sections of the graph of \( y = f(x) \) which are below the \( x \)-axis.

(c) From your knowledge of the definition of absolute value (modulus) explain the result that you have observed.

2 (a) Compare the graphs of the following pairs of functions, by sketching each pair on the same axes.
   (i) \( y = x + 1 \) and \( y = |x| + 1 \)
   (ii) \( y = x^2 - 3x + 2 \) and \( y = |x^2 - 3x| + 2 \)
   (iii) \( y = \sin x - 0.5 \) and \( y = \sin |x| - 0.5 \)

(b) For each of the absolute value graphs what symmetry do you notice?

(c) From your knowledge of the definition of modulus, explain the result that you have observed.
1 Investigating logarithms (continued)

(e) Copy and complete the following table by choosing your own numbers. An example has been given.

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>3 log 6</td>
<td>log 6^3</td>
<td>2.3345</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(d) Find a general pattern for \( n \log x \).

(e) Can you suggest why this is true?

4 Consider the function \( y = \log x \).

(a) When \( x = 1 \), find the value of \( y \).

(b) Where does the curve cut the x-axis?

(c) Can \( x = 0 \)? Can \( x < 0 \)? Use your calculator to check your answers.

(d) State the restricted domain of the function.

(e) Copy and complete the following table of values.

<table>
<thead>
<tr>
<th>( x )</th>
<th>0.000001</th>
<th>0.00001</th>
<th>0.0001</th>
<th>0.01</th>
<th>0.1</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y = \log x )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(f) What can you say about the y-axis?

(g) Copy and complete the following table of values.

<table>
<thead>
<tr>
<th>( x )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y = \log x )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(h) Using a scale of 1 cm to represent 1 unit on the x-axis and 2 cm to represent 1 unit on the y-axis, draw the curve \( y = \log x \).
1 Investigating logarithms (continued)

2 (a) Copy and complete the following table using your calculator. Give your answers correct to four decimal places.

<table>
<thead>
<tr>
<th>Expression</th>
<th>Answer</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \log 12 - \log 3 )</td>
<td>.7782</td>
</tr>
<tr>
<td>( \log 4 )</td>
<td></td>
</tr>
<tr>
<td>( \log 50 - \log 2 )</td>
<td></td>
</tr>
<tr>
<td>( \log 25 )</td>
<td></td>
</tr>
<tr>
<td>( \log 7 - \log 5 )</td>
<td></td>
</tr>
<tr>
<td>( \log 1.4 )</td>
<td></td>
</tr>
<tr>
<td>( \log 3 - \log 4 )</td>
<td></td>
</tr>
<tr>
<td>( \log 0.75 )</td>
<td></td>
</tr>
<tr>
<td>( \log 20 - \log 40 )</td>
<td></td>
</tr>
<tr>
<td>( \log \frac{1}{2} )</td>
<td></td>
</tr>
</tbody>
</table>

(b) Do you see any pattern? Describe it in your own words.

(c) Copy and complete the following table by choosing your own numbers. An example has been given.

<table>
<thead>
<tr>
<th>Expression</th>
<th>Answer</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \log 6 - \log 2 )</td>
<td></td>
</tr>
<tr>
<td>( \log 3 )</td>
<td>0.4771</td>
</tr>
</tbody>
</table>

(d) Find a general pattern for \( \log x - \log y \).

(e) Can you suggest why this is true?

3 (a) Copy and complete the following table using your calculator. Give your answers correct to four decimal places.

<table>
<thead>
<tr>
<th>Expression</th>
<th>Answer</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 4 \log 2 )</td>
<td>1.2041</td>
</tr>
<tr>
<td>( 5 \log 6 )</td>
<td></td>
</tr>
<tr>
<td>( \log 6^5 )</td>
<td></td>
</tr>
<tr>
<td>( \frac{1}{2} \log 4 )</td>
<td></td>
</tr>
<tr>
<td>( \log 4^{\frac{1}{2}} )</td>
<td></td>
</tr>
<tr>
<td>( \log 7 )</td>
<td></td>
</tr>
<tr>
<td>( \log 7^{0.5} )</td>
<td></td>
</tr>
<tr>
<td>( -3 \log 5 )</td>
<td></td>
</tr>
<tr>
<td>( \log 5^3 )</td>
<td></td>
</tr>
</tbody>
</table>

(b) Do you see any pattern? Describe it in your own words.
1 Investigating logarithms

Work done on this assignment will be assessed against criteria A, B, C, D and E. While the use of a graphic display calculator may be helpful, the use of technology is not specifically required, nor will it be assessed.
(NOTE: this would be considered a type II assignment if students had already completed a study of logarithms, and only the first four criteria should be assessed.)

1 (a) Copy and complete the following table using your calculator. Give your answers correct to four decimal places.

<table>
<thead>
<tr>
<th>Expression</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>log 2 + log 3</td>
<td>0.7782</td>
</tr>
<tr>
<td>log 6</td>
<td></td>
</tr>
<tr>
<td>log 3 + log 7</td>
<td></td>
</tr>
<tr>
<td>log 21</td>
<td></td>
</tr>
<tr>
<td>log 4 + log 20</td>
<td></td>
</tr>
<tr>
<td>log 80</td>
<td></td>
</tr>
<tr>
<td>log 0.2 + log 11</td>
<td></td>
</tr>
<tr>
<td>log 2.2</td>
<td></td>
</tr>
<tr>
<td>log 0.3 + log 0.4</td>
<td></td>
</tr>
<tr>
<td>log 0.12</td>
<td></td>
</tr>
</tbody>
</table>

(b) Do you see any pattern? Describe it in your own words.

(c) Copy and complete the following table by choosing your own numbers. An example has been given.

<table>
<thead>
<tr>
<th>Expression</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>log 5 + log 4</td>
<td></td>
</tr>
<tr>
<td>log 20</td>
<td>1.3010</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(d) Find a general pattern for \( \log x + \log y \).

(e) Can you suggest why this is true?
5 Areas under curves

Type I

Work done in this assignment will be assessed against criteria A, B, C, D, E and F. You are therefore expected to use a graphic calculator and/or computer for this assignment.

In this assignment, you will be calculating areas under the curves of power functions, i.e., functions of the form \( y = kx^n \), where \( k \neq 0 \), and \( n \in \mathbb{Z} \). This assignment introduces the use of the definite integral to find areas under curves. It assumes a knowledge of summation and limiting processes.

Show all your working and calculations clearly, and where possible, give your answer as a fraction in its simplest form.

1. (a) Sketch the graph of \( y = x^0 \), where \( x \in [0, 1] \).
   (b) Shade the area under the curve.
   (c) What geometrical shape is this area?
   (d) Calculate the area under the curve.

2. (a) Sketch the graph of \( y = x^1 \), where \( x \in [0, 1] \).
   (b) Shade the area under the curve.
   (c) What geometrical shape is this area?
   (d) Calculate the area under the curve.

3. (a) Sketch the graph of \( y = x^2 \), where \( x \in [0, 1] \).
    It will be necessary to estimate this area by dividing the region into parts which resemble shapes whose areas you can approximate.
    (b) Draw a vertical line at \( x = 0.5 \), to divide the region into two sections.
        (i) Estimate as accurately as possible the area of each section.
        (ii) Calculate the area under the curve.
    (c) Draw vertical lines at \( x = 0.25 \) and \( x = 0.75 \).
        (i) Estimate as accurately as possible the area of each section.
        (ii) Calculate the area under the curve.
    (d) Draw vertical lines at \( x = 0.2; x = 0.4; x = 0.6 \) and \( x = 0.8 \).
        Calculate the area under the curve.
    (e) Compare your answers to parts (b), (c) and (d) above. If you divide this region into even smaller sections and summed the areas, what do you think would happen to the total area? What value is the area \( A \) approaching? Express this in the form
    \[ A = \lim_{n \to \infty} \sum_{i=0}^{1} x^2 \delta x, \]
    where \( c \) is the number of vertical lines drawn.
5 Areas under curves (continued)

4  
(a) Sketch the graph of \( y = x^3 \), where \( x \in [0, 1] \)

(b) Calculate the area under the curve. Be sure that your non-calculator based method is clearly explained. (You can use the calculator to get points and measurements, but you can’t use it to calculate the answer for you by pressing the right buttons).

5  
The area \( A \) under the curve \( y = f(x) \) between \( x = 0 \) and \( x = 1 \) may be written as
\[
A = \int_{0}^{1} f(x) \, dx.
\]

(a) Write your answers to parts 1 – 4 above in this form.

(i) \( \int_{0}^{1} x^0 \, dx = \)  
(ii) \( \int_{0}^{1} x^1 \, dx = \)  
(iii) \( \int_{0}^{1} x^2 \, dx = \)  
(iv) \( \int_{0}^{1} x^3 \, dx = \)

(b) Estimate the areas under the following curves, where \( x \in [0, 1] \), then use your calculator to work out the actual area. Copy and complete the table to show your results.

<table>
<thead>
<tr>
<th>Function ( f(x) )</th>
<th>Estimate of area under the curve</th>
<th>Actual area under the curve</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y = x^5 )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( y = x^6 )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( y = x^{10} )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( y = x^{40} )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( y = x^{100} )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

6  
Make a conjecture about the area under the curve between \( x = 0 \) and \( x = 1 \), for \( k = 1 \). Check your result for different values of \( n \). Is your result valid for all values of \( n \)?
5 Areas under curves (continued)

Is your conjecture valid if you change your limits? Use your calculator to evaluate the areas represented below.

<table>
<thead>
<tr>
<th></th>
<th>( a ) ( \int_0^2 x^2 , dx = )</th>
<th>( f ) ( \int_1^1 x^3 , dx = )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( b ) ( \int_1^2 x^2 , dx = )</td>
<td>( g ) ( \int_{-1}^1 2x^3 , dx = )</td>
</tr>
<tr>
<td></td>
<td>( c ) ( \int_2^3 x^2 , dx = )</td>
<td>( h ) ( \int_{-1}^2 x^4 , dx = )</td>
</tr>
<tr>
<td></td>
<td>( d ) ( \int_{-1}^2 x^2 , dx = )</td>
<td>( i ) ( \int_0^1 x^{-1} , dx = )</td>
</tr>
<tr>
<td></td>
<td>( e ) ( \int_1^2 3x^2 , dx = )</td>
<td>( j ) ( \int_1^2 x^{-1} , dx = )</td>
</tr>
</tbody>
</table>

Try and modify your conjecture to take account of these results. Justify your modifications if possible, i.e. try and find a general formula for \( \int_a^b x^n \, dx \).
7 Fish pond

Work done on this assignment will be assessed against criteria A, B, C and D. While use of a graphic display calculator may be helpful, the use of technology is not specifically required, nor will it be assessed.

You have recently acquired a fish pond in which you plan to raise fish to sell at the local market. To do this efficiently and cost effectively, you want to determine the carrying capacity or maximum population that the pond is capable of sustaining in the long run. You have the following information:

- a scaled map of the pond (see below)

• an average depth of 1.5 meters
• one fish requires 37 m$^3$ to grow to its maximum size and does so in one year.

1 Estimate the surface area of the pond by each of the following methods.

(a) Use the sum of the areas of the three different triangles: $\triangle ABC$, $\triangle ACD$ and $\triangle ADE$

(b) Divide the pond into rectangles and sum the areas.

(c) Use the area of trapeziums (trapezoids).
7 Fish pond (continued)

2 (a) Which of the three above methods do you consider to be the most accurate? Why?
    (b) Try to find a more accurate estimate, carefully documenting your method.

3 (a) Find the volume of water in the pond using your most accurate estimate.
    (b) What is the carrying capacity of your pond?

4 Assume that the fish population grows according to the equation

\[ p = \frac{m}{1 + ae^{-kt}} \]

where \( p \) is the initial population, \( m \) is the carrying capacity, \( k = 0.9 \) and \( t \) is the number of years.

(a) Given that you stock the pond with 1000 fish initially, determine the value of \( a \).

(b) How many years will it take for your pond to reach its carrying capacity?

(c) What would you do to halve the time in part (b)?

(d) Can you harvest 1000 fish each year for the next three years? Give reasons for your answer.

(e) Can you harvest 2000 fish each year for the next three years? Give reasons for your answer.
1 Continued fractions

Description
Consider the continued fraction below.

\[ 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \ldots}}}} \]

We can consider this "infinite fraction" as a sequence of terms, \( t_n \), where

\[ t_1 = 1 + 1 \]

\[ t_2 = 1 + \frac{1}{1 + 1} \]

\[ t_3 = 1 + \frac{1}{1 + \frac{1}{1 + 1}} \]

\[ \ldots \]

Method
1. Determine a generalized formula for \( t_{n+1} \) in terms of \( t_n \).
2. Compute the decimal equivalents of the first 10 terms. Enter the terms into a data table and plot the relation between \( n \) and \( t_n \) using a GDC or computer. Provide printed output of your plot. What do you notice? What does this suggest about the value of \( t_n - t_{n+1} \) as \( n \) gets very large?
3. What problems arise when you try to determine the 200th term?
4. Use the results of step 1 and step 2 to establish an exact value for the continued fraction.
5. Now consider another continued fraction.

\[
2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{\ldots}}}}
\]

Repeat steps 1 to 4 using this continued fraction.

6. Now consider the general continued fraction.

\[
k + \frac{1}{k + \frac{1}{k + \frac{1}{k + \frac{1}{\ldots}}}}
\]

By considering other values of \(k\), determine a generalized statement for the exact value of any such continued fraction. For which values of \(k\) does your generalized statement hold true? How do you know? Provide evidence to support your answer.
2  The Koch snowflake

Description
In 1904 Helge von Koch identified a fractal that appeared to model the snowflake. The fractal is built by starting with an equilateral triangle and removing the inner third of each side, building another equilateral triangle at the location where the side was removed, and then repeating the process indefinitely. The process is illustrated clearly below, showing the original triangle at stage 0 and the resulting figures after one, two and three iterations.

![Koch Snowflake Stages](image)

**Method**
Let $N_n$ = the number of sides, $l_n$ = the length of a single side, $P_n$ = the length of the perimeter and $A_n$ = the area of the snowflake, at the $n^{th}$ stage.

1. Using an initial side length of 1, create a table that shows the values of $N_n$, $l_n$, $P_n$ and $A_n$ for $n = 0$, 1, 2 and 3. Use exact values in your results. Explain the relationship between successive terms in the table for each quantity $N_n$, $l_n$, $P_n$ and $A_n$.

2. Using a GDC or a suitable graphing software package, create graphs of the four sets of values plotted against the value of $n$. Provide separate printed output for each graph.

3. For each of the graphs above, develop a statement in terms of $n$ that generalizes the behaviour shown in its graph. Explain how you arrived at your generalizations. Verify that your generalizations apply consistently to the sets of values produced in the table.

4. Investigate what happens at $n = 4$. Use your conjectures from step 3 to obtain values for $N_4$, $l_4$, $P_4$ and $A_4$. Now draw a large diagram of one “side” (that is, one side of the original triangle that has been transformed) of the fractal at stage 4 and clearly verify your predictions.

5. Calculate values for $N_6$, $l_6$, $P_6$ and $A_6$. You need not verify these answers.

6. Write down successive values of $A_0$ in terms of $A_0$. What pattern emerges?

7. Explain what happens to the perimeter and area as $n$ gets very large. What conclusion can you make about the area as $n \to \infty$? Comment on your results.
3 Matrix powers

**Method**

1. Consider the matrix $\mathbf{M} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$.
   
   Calculate $\mathbf{M}^n$ for $n = 2, 3, 4, 5, 10, 20, 50$.
   
   Describe in words any pattern you observe.
   
   Use this pattern to find a general expression for the matrix $\mathbf{M}^n$ in terms of $n$.

2. Consider the matrices $\mathbf{P} = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$ and $\mathbf{S} = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}$.

   \[
   \mathbf{P}^2 = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}^2 = \begin{pmatrix} 10 & 6 \\ 6 & 10 \end{pmatrix} = 2 \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix};
   \]
   \[
   \mathbf{S}^2 = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}^2 = \begin{pmatrix} 20 & 16 \\ 16 & 20 \end{pmatrix} = 2 \begin{pmatrix} 10 & 8 \\ 8 & 10 \end{pmatrix}
   \]

   Calculate $\mathbf{P}^n$ and $\mathbf{S}^n$ for other values of $n$ and describe any pattern(s) you observe.

3. Now consider matrices of the form $\begin{pmatrix} k+1 & k-1 \\ k-1 & k+1 \end{pmatrix}$

   Steps 1 and 2 contain examples of these matrices for $k = 1, 2$ and $3$.

   Consider other values of $k$, and describe any pattern(s) you observe.

   Generalize these results in terms of $k$ and $n$.

4. Use technology to investigate what happens with further values of $k$ and $n$. State the scope or limitations of $k$ and $n$.

5. Explain why your results hold true in general.
### Mathematics SL: The portfolio

**Teacher’s record**

<table>
<thead>
<tr>
<th>Type I tasks</th>
<th>Form A</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Title of task:</strong> Matrix powers</td>
<td><strong>Type:</strong> I  II</td>
</tr>
<tr>
<td><strong>Date set:</strong> 12/03/05</td>
<td><strong>Date submitted:</strong> 20/03/05</td>
</tr>
</tbody>
</table>

#### Syllabus topics covered

4.2, 4.3

#### Background information

**Purpose of the task**

To provide further practice and appreciation of powers of matrices. In considering scope and limitations I hoped that students would discover that only integer powers are defined.

#### Previous exposure to relevant concepts/skills

Students have multiplied matrices but not considered powers of matrices.

#### Previous exposure to relevant terminology

Students are familiar with relevant terminology except inverse of a matrix.

#### Available technology

Students each have a Texas Instruments -83 plus™, and Windows Graphlink® software is always available in the computer lab. Students also have access to Autograph® software.

#### Teacher expectations regarding technology

For this task I did not expect students to provide printout; handwritten matrices would be sufficient. I did expect that they would acknowledge where they made use of GDC or other technology. I was able to see use of calculators in initial and final classroom sessions on the task.
4 Derivatives of sine functions

Please note that this task is only suitable for use before students have studied derivatives of sine functions and the chain rule. It is a way of introducing the topics graphically.

Method

1. Investigate the derivative of the function \( f(x) = \sin x \).
   - Graph the function \( f(x) = \sin x \) for \(-2\pi \leq x \leq 2\pi\).
   - Based only on this graph, describe as carefully and fully as you can, the behaviour of the gradient of the function on the given domain. Include a sketch of the graph of \( y = f'(x) \).
   - Make a conjecture for the derived function.
   - Use your calculator to test your conjecture graphically. You may find the nDeriv function useful. Explain your method and your findings. Modify your conjecture if necessary.
   - Use your calculator to verify your conjecture numerically. Explain your method and your findings. Modify your conjecture if necessary.

2. Investigate the derivatives of functions of the form \( g(x) = a \sin x \) in a similar way.
   - Consider several different values of \( a \).
   - Make a conjecture for \( g'(x) \).
   - Test the conjecture with further examples.
   - State for what values of \( a \) the conjecture holds.

3. Investigate the derivatives of functions of the form \( h(x) = \sin bx \) in a similar way.

4. Investigate the derivatives of functions of the form \( j(x) = \sin(x + c) \) in a similar way.

5. Use your results to steps 1 to 4 to make a conjecture for the derivative of \( k(x) = a \sin b(x + c) \). Choose a value for \( a, b \) and for \( c \). Verify your conjecture for this particular case.

6. Consider \( m(x) = \sin^2 x \). Investigate the derivative of this function as before and show that it can be written as \( m'(x) = 2\sin x \cos x \).
<table>
<thead>
<tr>
<th><strong>Mathematics SL: The portfolio</strong></th>
<th>Form A</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Teacher’s record</strong></td>
<td></td>
</tr>
<tr>
<td><strong>Title of task:</strong> Derivatives of sine functions</td>
<td><strong>Type:</strong> I, II</td>
</tr>
<tr>
<td><strong>Date set:</strong> 12/03/05</td>
<td><strong>Date submitted:</strong> 20/03/05</td>
</tr>
<tr>
<td><strong>Syllabus topics covered</strong></td>
<td></td>
</tr>
<tr>
<td>7.1, 7.2</td>
<td></td>
</tr>
<tr>
<td><strong>Background information</strong></td>
<td></td>
</tr>
<tr>
<td>To introduce and make plausible the derivatives of ( \sin x ) and simple composites with ( \sin x ).</td>
<td></td>
</tr>
<tr>
<td><strong>Previous exposure to relevant concepts/skills</strong></td>
<td></td>
</tr>
<tr>
<td>Students are familiar with the derivative as a function to describe gradient, with transformations of graphs of trig functions and with some double angle formulae.</td>
<td></td>
</tr>
<tr>
<td><strong>Previous exposure to relevant terminology</strong></td>
<td></td>
</tr>
<tr>
<td>Students are familiar with all relevant terminology.</td>
<td></td>
</tr>
<tr>
<td><strong>Available technology</strong></td>
<td></td>
</tr>
<tr>
<td>Students each have a Texas Instruments -83 plus*, and Windows Graphlink® software is always available in the computer lab. Students also have access to Autograph® software.</td>
<td></td>
</tr>
<tr>
<td><strong>Teacher expectations regarding technology</strong></td>
<td></td>
</tr>
<tr>
<td>Students have previously used nDeriv to graph the gradient function and are competent in use of tables. These are the approaches expected. Calculator screen dumps or graphs printed out from Autograph® are expected in support of the work done.</td>
<td></td>
</tr>
</tbody>
</table>
5 Stopping distances

Description
When a driver stops her car, she must first think to apply the brakes. Then the brakes must actually stop the vehicle.

The table below lists average times for these processes at various speeds.

<table>
<thead>
<tr>
<th>Speed (kmh⁻¹)</th>
<th>Thinking distance (m)</th>
<th>Braking distance (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>32</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>48</td>
<td>9</td>
<td>14</td>
</tr>
<tr>
<td>64</td>
<td>12</td>
<td>24</td>
</tr>
<tr>
<td>80</td>
<td>15</td>
<td>38</td>
</tr>
<tr>
<td>96</td>
<td>18</td>
<td>55</td>
</tr>
<tr>
<td>112</td>
<td>21</td>
<td>75</td>
</tr>
</tbody>
</table>

In this task you will develop individual functions that model the relationships between speed and thinking distance, as well as speed and braking distance. You will also develop a model for the relationship between speed and overall stopping distance.

Method
1. Use a GDC or graphing software to create two data plots: speed versus thinking distance and speed versus braking distance. Describe your results.
2. Using your knowledge of functions, develop functions that model the behaviours noted in step 1. Explain your work.
3. The overall stopping distance is obtained from adding the thinking distance to the braking distance. Create a data table of speed and overall stopping distance. Graph this data and describe the results.
4. Develop a function that models the relationship between speed and overall stopping distance. How is this function related to the functions obtained in step 2?
5. Overall stopping distances for other speeds are given below. Discuss how your model fits this data, and what modifications might be necessary.

<table>
<thead>
<tr>
<th>Speed (kmh⁻¹)</th>
<th>Stopping distance (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>2.5</td>
</tr>
<tr>
<td>40</td>
<td>17</td>
</tr>
<tr>
<td>90</td>
<td>65</td>
</tr>
<tr>
<td>160</td>
<td>180</td>
</tr>
</tbody>
</table>
6 Sunrise over New York

**Description**

The table shows times of sunrise over New York at weekly intervals during 2003, starting on 1 January. All times are Eastern Standard Time in hours and minutes.

<table>
<thead>
<tr>
<th>Time</th>
<th>Week</th>
<th>Time</th>
<th>Week</th>
<th>Time</th>
<th>Week</th>
</tr>
</thead>
<tbody>
<tr>
<td>07.20</td>
<td>1</td>
<td>04.34</td>
<td>21</td>
<td>06.06</td>
<td>41</td>
</tr>
<tr>
<td>07.20</td>
<td>2</td>
<td>04.29</td>
<td>22</td>
<td>06.14</td>
<td>42</td>
</tr>
<tr>
<td>07.18</td>
<td>3</td>
<td>04.26</td>
<td>23</td>
<td>06.22</td>
<td>43</td>
</tr>
<tr>
<td>07.14</td>
<td>4</td>
<td>04.24</td>
<td>24</td>
<td>06.30</td>
<td>44</td>
</tr>
<tr>
<td>07.09</td>
<td>5</td>
<td>04.26</td>
<td>25</td>
<td>06.39</td>
<td>45</td>
</tr>
<tr>
<td>07.02</td>
<td>6</td>
<td>04.29</td>
<td>26</td>
<td>06.47</td>
<td>46</td>
</tr>
<tr>
<td>06.54</td>
<td>7</td>
<td>04.33</td>
<td>27</td>
<td>06.55</td>
<td>47</td>
</tr>
<tr>
<td>06.45</td>
<td>8</td>
<td>04.38</td>
<td>28</td>
<td>07.02</td>
<td>48</td>
</tr>
<tr>
<td>06.35</td>
<td>9</td>
<td>04.44</td>
<td>29</td>
<td>07.08</td>
<td>49</td>
</tr>
<tr>
<td>06.24</td>
<td>10</td>
<td>04.50</td>
<td>30</td>
<td>07.14</td>
<td>50</td>
</tr>
<tr>
<td>06.13</td>
<td>11</td>
<td>04.57</td>
<td>31</td>
<td>07.18</td>
<td>51</td>
</tr>
<tr>
<td>06.01</td>
<td>12</td>
<td>05.04</td>
<td>32</td>
<td>07.18</td>
<td>52</td>
</tr>
<tr>
<td>05.50</td>
<td>13</td>
<td>05.11</td>
<td>33</td>
<td></td>
<td></td>
</tr>
<tr>
<td>05.38</td>
<td>14</td>
<td>05.17</td>
<td>34</td>
<td></td>
<td></td>
</tr>
<tr>
<td>05.27</td>
<td>15</td>
<td>05.24</td>
<td>35</td>
<td></td>
<td></td>
</tr>
<tr>
<td>05.16</td>
<td>16</td>
<td>05.31</td>
<td>36</td>
<td></td>
<td></td>
</tr>
<tr>
<td>05.06</td>
<td>17</td>
<td>05.38</td>
<td>37</td>
<td></td>
<td></td>
</tr>
<tr>
<td>04.56</td>
<td>18</td>
<td>05.45</td>
<td>38</td>
<td></td>
<td></td>
</tr>
<tr>
<td>04.47</td>
<td>19</td>
<td>05.52</td>
<td>39</td>
<td></td>
<td></td>
</tr>
<tr>
<td>04.40</td>
<td>20</td>
<td>05.59</td>
<td>40</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Source: http://www.usno.navy.mil/
Method

1. Use a graphing package or spreadsheet to draw a graph of this data.

2. What type of function might be suitable to model this data? Explain what assumptions you are making.

3. Use your knowledge of the graphs of such functions to find a suitable function that models the behaviour. Identify any variables and parameters clearly, and explain how you determined them. Comment on how well your function fits the data.

4. Use a regression tool to find the best fit function. Comment on any differences from the function you found in step 3.

5. Compare the usefulness of the two models to:
   (a) someone planning a daybreak run
   (b) someone programming switching off the street-lighting.

6. How would each of these change if you travelled 1000km:
   (a) north
   (b) south
   (c) west?
   Use the Internet to find data to support your answers.

7. This graph shows the corresponding times of sunset.

Sunset over New York

Use this information together with one of the models for sunrise to find estimates for:
   (a) the length of the shortest day
   (b) the approximate dates between which the day is more than 12 hours long.
7 Tide modelling

Description
The Bay of Fundy in Nova Scotia, Canada is deemed to have the greatest average change in tide height in the world. In the table below data is presented from 27 December 2003 using Atlantic Standard Time (AST). The heights were taken at Grindstone Island.

<table>
<thead>
<tr>
<th>Time (AST)</th>
<th>00.00</th>
<th>01.00</th>
<th>02.00</th>
<th>03.00</th>
<th>04.00</th>
<th>05.00</th>
<th>06.00</th>
<th>07.00</th>
<th>08.00</th>
<th>09.00</th>
<th>10.00</th>
<th>11.00</th>
</tr>
</thead>
<tbody>
<tr>
<td>Height (m)</td>
<td>7.5</td>
<td>10.2</td>
<td>11.8</td>
<td>12.0</td>
<td>10.9</td>
<td>8.9</td>
<td>6.3</td>
<td>3.6</td>
<td>1.6</td>
<td>0.9</td>
<td>1.8</td>
<td>4.0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Time (AST)</th>
<th>12.00</th>
<th>13.00</th>
<th>14.00</th>
<th>15.00</th>
<th>16.00</th>
<th>17.00</th>
<th>18.00</th>
<th>19.00</th>
<th>20.00</th>
<th>21.00</th>
<th>22.00</th>
<th>23.00</th>
</tr>
</thead>
<tbody>
<tr>
<td>Height (m)</td>
<td>6.9</td>
<td>9.7</td>
<td>11.6</td>
<td>12.3</td>
<td>11.6</td>
<td>9.9</td>
<td>7.3</td>
<td>4.5</td>
<td>2.1</td>
<td>0.7</td>
<td>0.8</td>
<td>2.4</td>
</tr>
</tbody>
</table>

Source: http://www.lau.chs.shc.dfo-mpo.gc.ca

In this task you will develop a model function for the relationship between time of day and the height of the tide. Consider carefully the expectations of a modelling task as you complete your work.

Method
1. Using a GDC or graphing software, plot the graph of time against height. Describe the result.
2. Use your knowledge of functions to develop a function that models the behaviour noted in the graph. Describe any variables, parameters or constraints for the model. Explain clearly how you established the value of any parameters.
3. Draw a graph of your function on the same set of axes as the graph in step 1. How well does the function fit the data?
4. Modify the function to create a better fit. Describe the issues you had to consider.
5. Good sailors will launch their boats on an outgoing tide (that is when the tide is going out). Use your model to determine the times between which a good sailor would have launched a boat on 27 December 2003.
6. Use the regression feature of your GDC or software to develop a best-fit function for this data. Compare this function with the one you developed analytically.
7. The table below lists the tide heights for 28 December 2003. Does your function fit these data? What modifications are needed? Confirm that your modified model fits the data.

<table>
<thead>
<tr>
<th>Time (AST)</th>
<th>00.00</th>
<th>01.00</th>
<th>02.00</th>
<th>03.00</th>
<th>04.00</th>
<th>05.00</th>
<th>06.00</th>
<th>07.00</th>
<th>08.00</th>
<th>09.00</th>
<th>10.00</th>
<th>11.00</th>
</tr>
</thead>
<tbody>
<tr>
<td>Height (m)</td>
<td>5.0</td>
<td>7.9</td>
<td>10.2</td>
<td>11.6</td>
<td>11.6</td>
<td>10.5</td>
<td>8.5</td>
<td>6.0</td>
<td>3.5</td>
<td>1.7</td>
<td>1.2</td>
<td>2.2</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Time (AST)</th>
<th>12.00</th>
<th>13.00</th>
<th>14.00</th>
<th>15.00</th>
<th>16.00</th>
<th>17.00</th>
<th>18.00</th>
<th>19.00</th>
<th>20.00</th>
<th>21.00</th>
<th>22.00</th>
<th>23.00</th>
</tr>
</thead>
<tbody>
<tr>
<td>Height (m)</td>
<td>4.4</td>
<td>7.2</td>
<td>9.7</td>
<td>11.3</td>
<td>11.8</td>
<td>11.1</td>
<td>9.4</td>
<td>7.0</td>
<td>4.4</td>
<td>2.2</td>
<td>1.0</td>
<td>1.3</td>
</tr>
</tbody>
</table>
8 Modelling the amount of a drug in the bloodstream

Description
The graph below records the amount of a drug for treating malaria in the bloodstream over the 10 hours following an initial dose of 10μg.

![Graph showing amount of drug in the bloodstream over time.]

It seems that the rate of decrease of the drug is approximately proportional to the amount remaining.

Method

Part A
1. Use this information to help you find a suitable function to model this data.
2. Draw a graph of your function and compare your graph to the one above.
3. Comment on the suitability of the model.

Part B
A patient is instructed to take 10μg of this drug every six hours.

1. Sketch a diagram to show the amount of the drug in the bloodstream over a 24-hour period and state any assumptions made.
2. Use your GDC or graphing software and your model from part A to draw an accurate graph to represent this situation.
3. State the maximum and minimum amounts during this period.
4. Describe what would happen to these values over the next week if:
   (a) no further doses are taken
   (b) doses continue to be taken every six hours.
INFINITE SURDS

The following expression is an example of an infinite surd.

\[ \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \ldots}}}} \]

Consider this surd as a sequence of terms \( a_n \) where:

\[ a_1 = \sqrt{1 + \sqrt{1}} \]
\[ a_2 = \sqrt{1 + \sqrt{1 + \sqrt{1}}} \]
\[ a_3 = \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1}}}} \text{ etc.} \]

Find a formula for \( a_{n+1} \) in terms of \( a_n \).

Calculate the decimal values of the first ten terms of the sequence. Using technology, plot the relation between \( n \) and \( a_n \). Describe what you notice. What does this suggest about the value of \( a_n - a_{n+1} \) as \( n \) gets very large? Use your results to find the exact value for this infinite surd.

Consider another infinite surd \( \sqrt{2 + \sqrt{2 + \sqrt{2 + 
\ldots}} \text{ where the first term is } \sqrt{2 + \sqrt{2}} \). Repeat the entire process above to find the exact value for this surd.

Now consider the general infinite surd \( \sqrt{k + \sqrt{k + \sqrt{k + \sqrt{k + \ldots}}}} \text{ where the first term is } \sqrt{k + \sqrt{k}} \). Find an expression for the exact value of this general infinite surd in terms of \( k \).

The value of an infinite surd is not always an integer.

Find some values of \( k \) that make the expression an integer. Find the general statement that represents all the values of \( k \) for which the expression is an integer.

Test the validity of your general statement using other values of \( k \).

Discuss the scope and/or limitations of your general statement.

Explain how you arrived at your general statement.
LOGARITHM BASES

Consider the following sequences. Write down the next two terms of each sequence.

\[ \log_2 8, \log_4 8, \log_8 8, \log_{16} 8, \log_{32} 8, \ldots \]

\[ \log_3 81, \log_9 81, \log_{27} 81, \log_{81} 81, \ldots \]

\[ \log_5 25, \log_{25} 25, \log_{125} 25, \log_{625} 25, \ldots \]

\[ \ldots \]

\[ \log_m m^k, \log_{m^2} m^k, \log_{m^3} m^k, \log_{m^4} m^k, \ldots \]

Find an expression for the \( n \)th term of each sequence. Write your expressions in the form \( \frac{p}{q} \), where \( p, q \in \mathbb{Z} \). Justify your answers using technology.

Now calculate the following, giving your answers in the form \( \frac{p}{q} \), where \( p, q \in \mathbb{Z} \)

\[ \log_4 64, \log_8 64, \log_{32} 64 \]

\[ \log_7 49, \log_{49} 49, \log_{343} 49 \]

\[ \log_5 125, \frac{\log_5 125}{125}, \frac{\log_5 125}{625} \]

\[ \log_8 512, \log_2 512, \log_{16} 512 \]

Describe how to obtain the third answer in each row from the first two answers. Create two more examples that fit the pattern above.

Let \( \log_a x = c \) and \( \log_b x = d \). Find the general statement that expresses \( \log_{ab} x \), in terms of \( c \) and \( d \).

Test the validity of your general statement using other values of \( a, b, \) and \( x \).

Discuss the scope and/or limitations of \( a, b, \) and \( x \).

Explain how you arrived at your general statement.
MATRIX BINOMIALS

SL TYPE I

Let $X = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $Y = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$. Calculate $X^2$, $X^3$, $X^4$; $Y^2$, $Y^3$, $Y^4$.

By considering integer powers of $X$ and $Y$, find expressions for $X^n$, $Y^n$, $(X+Y)^n$.

Let $A = aX$ and $B = bY$, where $a$ and $b$ are constants.

Use different values of $a$ and $b$ to calculate $A^2$, $A^3$, $A^4$; $B^2$, $B^3$, $B^4$.

By considering integer powers of $A$ and $B$, find expressions for $A^n$, $B^n$, $(A+B)^n$.

Now consider $M = \begin{pmatrix} a+b & a-b \\ a-b & a+b \end{pmatrix}$.

Show that $M = A + B$, and that $M^2 = A^2 + B^2$.

Hence, find the general statement that expresses $M^n$ in terms of $aX$ and $bY$.

Test the validity of your general statement by using different values of $a$, $b$, and $n$.

Discuss the scope and/or limitations of your general statement.

Use an algebraic method to explain how you arrived at your general statement.
SHADY AREAS

In this investigation you will attempt to find a rule to approximate the area under a curve (i.e. between the curve and the x-axis) using trapeziums (trapezoids).

Consider the function \( g(x) = x^2 + 3 \).

The diagram below shows the graph of \( g \). The area under this curve from \( x = 0 \) to \( x = 1 \) is approximated by the sum of the area of two trapeziums. Find this approximation.

Increase the number of trapeziums to five and find a second approximation for the area.

With the help of technology, create diagrams showing an increasing number of trapeziums. For each diagram, find the approximation for the area. What do you notice?

(This task continues on the following page)
Use the diagram below to find a general expression for the area under the curve of \( g \), from \( x = 0 \) to \( x = 1 \), using \( n \) trapeziums.

Use your results to develop the general statement that will estimate the area under any curve \( y = f(x) \) from \( x = a \) to \( x = b \) using \( n \) trapeziums. Show clearly how you developed your statement.

Consider the areas under the following three curves, from \( x = 1 \) to \( x = 3 \).

\[
\begin{align*}
y_1 &= \left( \frac{x}{2} \right)^2 \\
y_2 &= \frac{9x}{\sqrt{x^3 + 9}} \\
y_3 &= 4x^3 - 23x^2 + 40x - 18
\end{align*}
\]

Use your general statement, with eight trapeziums, to find approximations for these areas.

Find \( \int_1^3 \left( \frac{x}{2} \right)^2 \, dx \), \( \int_1^3 \left( \frac{9x}{\sqrt{x^3 + 9}} \right) \, dx \), \( \int_1^3 (4x^3 - 23x^2 + 40x - 18) \, dx \), and compare these answers with your approximations. Comment on the accuracy of your approximations.

Use other functions to explore the scope and limitations of your general statement. Does it always work? Discuss how the shape of a graph influences your approximation.
PARALLELS AND PARALLELOGRAMS

This task will consider the number of parallelograms formed by intersecting parallel lines.

Figure 1 below shows a pair of horizontal parallel lines and a pair of parallel transversals. One parallelogram \((A_1)\) is formed.

A third parallel transversal is added to the diagram as shown in Figure 2. Three parallelograms are formed: \(A_1, A_2, \text{ and } A_1 \cup A_2\).

We can go on drawing additional transversals and forming new parallelograms.

Show that six parallelograms are formed when a fourth transversal is added to Figure 2. List all these parallelograms, using set notation.

Repeat the process with 5, 6 and 7 transversals. Show your results in a table. Use technology to find a relation between the number of transversals and the number of parallelograms. Develop a general statement, and test its validity.

Next consider the number of parallelograms formed by three horizontal parallel lines intersected by parallel transversals. Develop and test another general statement for this case.

Now extend your results to \(m\) horizontal parallel lines intersected by \(n\) parallel transversals.

Display the results in a spreadsheet and use this to find the general statement for the overall pattern.

Test the validity of your statement.

Discuss its scope and/or limitations.

Explain how you arrived at this generalization.
BODY MASS INDEX

Body mass index (BMI) is a measure of one’s body fat. It is calculated by taking one’s weight (kg) and dividing by the square of one’s height (m).

The table below gives the median BMI for females of different ages in the US in the year 2000.

<table>
<thead>
<tr>
<th>Age (yrs)</th>
<th>BMI</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>16.40</td>
</tr>
<tr>
<td>3</td>
<td>15.70</td>
</tr>
<tr>
<td>4</td>
<td>15.30</td>
</tr>
<tr>
<td>5</td>
<td>15.20</td>
</tr>
<tr>
<td>6</td>
<td>15.21</td>
</tr>
<tr>
<td>7</td>
<td>15.40</td>
</tr>
<tr>
<td>8</td>
<td>15.80</td>
</tr>
<tr>
<td>9</td>
<td>16.30</td>
</tr>
<tr>
<td>10</td>
<td>16.80</td>
</tr>
<tr>
<td>11</td>
<td>17.50</td>
</tr>
<tr>
<td>12</td>
<td>18.18</td>
</tr>
<tr>
<td>13</td>
<td>18.70</td>
</tr>
<tr>
<td>14</td>
<td>19.36</td>
</tr>
<tr>
<td>15</td>
<td>19.88</td>
</tr>
<tr>
<td>16</td>
<td>20.40</td>
</tr>
<tr>
<td>17</td>
<td>20.85</td>
</tr>
<tr>
<td>18</td>
<td>21.22</td>
</tr>
<tr>
<td>19</td>
<td>21.60</td>
</tr>
<tr>
<td>20</td>
<td>21.65</td>
</tr>
</tbody>
</table>

(Source: [http://www.cdc.gov](http://www.cdc.gov))

Using technology, plot the data points on a graph. Define all variables used and state any parameters clearly.

What type of function models the behaviour of the graph? Explain why you chose this function. Create an equation (a model) that fits the graph.

On a new set of axes, draw your model function and the original graph. Comment on any differences. Refine your model if necessary.

Use technology to find another function that models the data. On a new set of axes, draw your model function and the function you found using technology. Comment on any differences.

Use your model to estimate the BMI of a 30-year-old woman in the US. Discuss the reasonableness of your answer.

Use the Internet to find BMI data for females from another country. Does your model also fit this data? If not, what changes would you need to make? Discuss any limitations to your model.
A fishing rod requires guides for the line so that it does not tangle and so that the line casts easily and efficiently. In this task, you will develop a mathematical model for the placement of line guides on a fishing rod.

The diagram shows a fishing rod with eight guides, plus a guide at the tip of the rod.

Leo has a fishing rod with overall length 230 cm. The table shown below gives the distances for each of the line guides from the tip of his fishing rod.

<table>
<thead>
<tr>
<th>Guide number (from tip)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Distance from tip (cm)</td>
<td>10</td>
<td>23</td>
<td>38</td>
<td>55</td>
<td>74</td>
<td>96</td>
<td>120</td>
<td>149</td>
</tr>
</tbody>
</table>

Define suitable variables, discuss parameters/constraints.

Using technology, plot the data points on a graph.

Using matrix methods or otherwise, find a quadratic function and a cubic function which model this situation. Explain the process you used. On a new set of axes, draw these model functions and the original data points. Comment on any differences.

Find a polynomial function which passes through every data point. Explain your choice of function, and discuss its reasonableness. On a new set of axes, draw this model function and the original data points. Comment on any differences.

Using technology, find one other function that fits the data. On a new set of axes, draw this model function and the original data points. Comment on any differences.

Mark has a fishing rod with overall length 300 cm. The table shown below gives the distances for each of the line guides from the tip of the Mark’s fishing rod.

<table>
<thead>
<tr>
<th>Guide number (from tip)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Distance from tip (cm)</td>
<td>10</td>
<td>22</td>
<td>34</td>
<td>48</td>
<td>64</td>
<td>81</td>
<td>102</td>
<td>124</td>
</tr>
</tbody>
</table>

How well does your quadratic model fit this new data? What changes, if any, would need to be made for that model to fit this data? Discuss any limitations to your model.

For final assessment in 2009 and 2010
CROWS DROPPING NUTS

Crows love nuts but their beaks are not strong enough to break some nuts open. To crack open the shells, they will repeatedly drop the nut on a hard surface until it opens.

The following table shows the average number of drops it takes to break open a large nut from varying heights.

<table>
<thead>
<tr>
<th>Height of drop (m)</th>
<th>1.7</th>
<th>2.0</th>
<th>2.9</th>
<th>4.1</th>
<th>5.6</th>
<th>6.3</th>
<th>7.0</th>
<th>8.0</th>
<th>10.0</th>
<th>13.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of drops</td>
<td>42.0</td>
<td>21.0</td>
<td>10.3</td>
<td>6.8</td>
<td>5.1</td>
<td>4.8</td>
<td>4.4</td>
<td>4.1</td>
<td>3.7</td>
<td>3.2</td>
</tr>
</tbody>
</table>

Using technology, plot the data points on a graph. Define all variables used and state any parameters clearly.

What type of function models the behaviour of the graph? Explain why you chose this function. Create an equation (a model) that fits the graph.

On a new set of axes, draw your model and the original graph. Comment on any differences. Refine your model if necessary.

Use technology to find another function that models the data. On a new set of axes, draw your model function and the function you found using technology. Comment on any differences.

The following tables show the average number of drops it takes to break open a medium nut, and a small nut, from varying heights.

Medium Nuts

<table>
<thead>
<tr>
<th>Height of drop (m)</th>
<th>1.5</th>
<th>2.0</th>
<th>3.0</th>
<th>4.0</th>
<th>5.0</th>
<th>6.0</th>
<th>7.0</th>
<th>8.0</th>
<th>10.0</th>
<th>15.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of drops</td>
<td>-</td>
<td>-</td>
<td>27.1</td>
<td>18.3</td>
<td>12.2</td>
<td>11.1</td>
<td>7.4</td>
<td>7.6</td>
<td>5.8</td>
<td>3.6</td>
</tr>
</tbody>
</table>

Small Nuts

<table>
<thead>
<tr>
<th>Height of drop (m)</th>
<th>1.5</th>
<th>2.0</th>
<th>3.0</th>
<th>4.0</th>
<th>5.0</th>
<th>6.0</th>
<th>7.0</th>
<th>8.0</th>
<th>10.0</th>
<th>15.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of drops</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>57.0</td>
<td>19.0</td>
<td>14.7</td>
<td>12.3</td>
<td>9.7</td>
<td>13.3</td>
<td>9.5</td>
</tr>
</tbody>
</table>

How well does your first model apply to nuts of different sizes? What changes, if any, need to be made to your model to fit the data for medium and small nuts? Discuss any limitations to your models.

For final assessment in 2009 and 2010
LOGAN’S LOGO

Note to teachers: *The size of the square is not critical until it is measured. Variations may result when copies of the task are made. Students should measure the diagram as it is presented. It will be very helpful to moderators if you include a copy of the task with any work selected for the sample.*

Logan has designed the logo below.

![Diagram of Logan's logo](image)

The diagram shows a square which is divided into three regions by two curves. The logo is the shaded region between the two curves. Logan wishes to develop mathematical functions that model these curves.

Using an appropriate set of axes, identify and record a number of data points on the curves which will allow you to develop model functions for them. Define all variables used and state any parameters clearly.

Using technology, plot these two sets of data points on a graph. What type of functions model the behaviour of the data? Explain why you chose these functions.

Find functions that represent the upper and lower curves forming the logo. Discuss any limitations.

Logan wishes to print T-shirts with the logo on the back. She must double the dimensions of the logo for this purpose. Describe how your functions must be modified.

Logan also wishes to print business cards. A standard business card is 9 cm by 5 cm. How must your functions be modified so that the logo extends from one end of the card to the other? Use technology to show the results.

What fraction of the area of the card does the logo occupy? Why might this be an important aspect of a business card?