Report: Topics covered include Structures and patterns

This sample is intended to inform the design of assessment instruments in the senior phase of learning. It highlights the qualities of student work and the match to the syllabus standards.

Criteria assessed

- Knowledge and procedures
- Modelling and problem solving
- Communication and justification

Assessment instrument

The response presented in this sample is in response to an assessment task.

Structures and patterns: Koch snowflake

Your report about the Koch snowflake will consist of two main sections.

In Section 1, you will consider a modified Koch snowflake.

It will provide opportunities to develop concepts associated with patterns in the modified structure.

Begin with an equilateral triangle and put a black triangle half the scale upside down inside it. This leaves 3 white triangles, and you can put a black triangle of half the scale upside down inside each of these. Continue doing this indefinitely, i.e. at each stage, put a black triangle of half the scale inside every white triangle in the diagram. The figure below illustrates the initial stages.

The figure below illustrates the initial stages.

![Stage 0](image1.png) ![Stage 1](image2.png) ![Stage 2](image3.png)

1. Find the area of the first black triangle, if the area of the main triangle is 1 unit².
2. Next:
   a. Determine the total area of the black triangles added at Stage 2.
   b. Express the answer to the above question as a fraction of the area of the first black triangle.
3. Explain why the area of the triangles added at the \( n \)th stage is the same fraction of the area of the triangles added at the \((n - 1)\)th stage.

4. Write down an expression for the area of all the black triangles added at the first \( n \) stages and then determine the area as \( n \to \infty \).

5. By observing the white triangles determine the fraction of the main triangle that is still white at Stage 1, Stage 2, Stage 3 and Stage \( n \).

6. Comment on the area of the white triangles as \( n \to \infty \).

In Section 2, you will investigate properties of the Koch snowflake itself.

The complete Koch snowflake can be constructed by starting with an equilateral triangle, then recursively altering each line segment as follows:

1. Divide the line segment into three segments of equal length.
2. Draw an equilateral triangle that has the middle segment from Step 1 as its base and points outward.
3. Remove the line segment that is the base of the triangle from Step 2.

After one iteration of this process, the resulting shape is the outline of a hexagram.

The Koch snowflake is the limit approached as the above steps are followed over and over again. The Koch curve originally described by von Koch is constructed with only one of the three sides of the original triangle. In other words, three Koch curves make a Koch snowflake.

![Koch Snowflake Diagram](image)

Stage: 1 2 3 4

This process can be continued indefinitely.

The report for Section 2 will consider the following:

If the total area added on when the Koch snowflake curve is developed indefinitely, show that it results in a finite area equal to \( \frac{8}{5} \) of the area of the initial triangle. Also show that the Koch snowflake curve has an infinite length, if the process outlined above is continued indefinitely.
### Instrument-specific criteria and standards

Student responses have been matched to instrument-specific criteria and standards; those which best describe the student work in this sample are shown below. For more information about the syllabus dimensions and standards descriptors, see [www.qsa.qld.edu.au1896.html#assessment](http://www.qsa.qld.edu.au1896.html#assessment).

<table>
<thead>
<tr>
<th>Knowledge and procedures</th>
<th>Standard A</th>
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<tbody>
<tr>
<td>The student work has the following characteristics:</td>
<td></td>
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<tr>
<td>• application of mathematical definitions, rules and procedures in routine and non-routine simple tasks through to routine complex tasks, in life-related and abstract situations</td>
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<tr>
<td>• numerical calculations, spatial sense and algebraic facility in routine and non-routine simple tasks through to routine complex tasks, in life-related and abstract situations</td>
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<th>Modelling and problem solving</th>
<th>Standard A</th>
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<tbody>
<tr>
<td>The student work has the following characteristics:</td>
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<tr>
<td>• use of problem-solving strategies to interpret, clarify and analyse problems to develop responses from routine simple tasks through to non-routine complex tasks in life-related and abstract situations</td>
<td></td>
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<tr>
<td>• use of data to synthesise mathematical models in simple through to complex situations</td>
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<tr>
<td>• investigation and evaluation of the validity of mathematical arguments including the analysis of results in the contexts of problems</td>
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</table>

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<th>Communication and justification</th>
<th>Standard A</th>
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<tr>
<td>The student work has the following characteristics:</td>
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<tr>
<td>• appropriate interpretation and use of mathematical terminology, symbols and conventions from simple through to complex and from routine through to non-routine, in life-related and abstract situations</td>
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<tr>
<td>• analysis and translation of information from one representation to another in life-related and abstract situations from simple through to complex and from routine through to non-routine</td>
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<tr>
<td>• use of mathematical reasoning to develop coherent, concise and logical sequences within a response from simple through to complex and in life-related and abstract situations using everyday and mathematical language</td>
<td></td>
</tr>
<tr>
<td>• coherent, concise and logical justification of procedures, decisions and results</td>
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</tbody>
</table>
Indicative response — Standard A

The annotations show the match to the instrument-specific standards.

Response

Section 1

1. Find the area of the first black triangle, if the area of the main triangle is 1 unit².

There are four triangles within the original white triangle the size of the black triangle and including the black triangle. There is just one black triangle; therefore, the black triangle is one quarter the area of the original area. This is because, as the black triangle is half the scale of the original white triangle, its dimensions are half that of the original, therefore the height of the black triangle is half, and the base is half.

The black triangle has a base which is half the length of the original triangle, and a perpendicular height which is also half that of the original. To determine the actual area of the black triangle, if the area of the original triangle is one unit², the ratio of the area of the black triangle to the original triangle must be determined. This is done below using the information stated above.

For the original white triangle use a base of 2 units, and perpendicular height of 4 units, these values to do not need to be of the proportion of an actual equilateral triangle, as this method holds true for all areas. Therefore, using the above information, the black triangle must have a base of 1 unit, and a perpendicular height of 2 units. The working for this example is given below, where W is the white triangle, B is the Black triangle, and A represents area.

\[
\text{Ratio } \frac{W}{B} = \frac{A_W}{A_B} = \frac{\text{Base}_W \times \text{Height}_W}{\text{Base}_B \times \text{Height}_B} \]

\[
= \frac{2 \times 4}{1 \times 2} = \frac{4}{2} = 2
\]

Therefore, it can be said, the area of the original white triangle is four times that of the first black triangle, and inversely, the area of the black triangle is one quarter that of the original white triangle. As such, the area of the black triangle is one quarter units², as the area of the original white triangle is one units², and the black triangle is one quarter the area of that triangle.
2.
   a. Determine the total area of black triangles added at Stage 2.

   The triangles added in stage two are one quarter the area of the black triangle in Stage 1.

   Therefore, the area of one of the black triangles being added in Stage 2 is one quarter of one quarter, which is one sixteenth. Working for this is shown below.

   \[
   \text{Triangle } A_{\text{stage 2}} = \frac{1}{4} \times \frac{1}{4} = \frac{1}{16} \text{ units}^2
   \]

   In stage two, the smallest triangle is 1/16 the size of the original white triangle. And there are three of the smaller triangles being added. Therefore, the total area added in stage two is 3/16 of the area of the original white triangle. This is shown below.

   \[
   \text{Area added in stage 2} = 3 \times \frac{1}{16} = \frac{3}{16} \text{ units}^2
   \]

   b. Express the answer to the above question as a fraction of the area of the first black triangle.

   The area of the first black triangle is one quarter that of the original white triangle. And the area of the black triangles added in stage two is 3/16 that of the original white triangle. The fraction of the area added in stage two to the area of the first black triangle is the quotient of the area of the first black triangle and the area of the triangles added in stage two. This is calculated below.

   \[
   \frac{\text{Stage 2}}{\text{Stage 1}} = \frac{\frac{3}{16}}{\frac{1}{4}} = \frac{3}{16} \times \frac{4}{1} = \frac{3}{4}
   \]

   Each of the smaller black triangles added at Stage 2 is one quarter the size of the black triangle in Stage 1. Therefore, as there are three smaller triangles being added, the amount of area added is three quarters of the first black triangle.
3. Explain why the area of the triangles added at the nth stage is the same fraction of the area of the first black triangle.

Every white triangle is the same size as the black triangles which were previously added. Each of the white triangles are then split into three more smaller white triangles, by one black triangle which is 1/4 the size of the first white triangle.

For every one black triangle, there is three white triangles the same size, so when the next stage of black triangles are added, three of them, which are 1/4 the size of the previous black triangle are added, thereby increasing the area by the product of three and one quarter, or three quarters.

The area of black being added is a geometric sequence with the common ratio \( r \) being, 3/4, and the total area of black at any stage is the sum of stages prior, and therefore is a geometric series.

4. Write down an expression for the area of all black triangles added at the first n stages and then determine the area as \( n \to \infty \).

The area of black triangles added at any stage is an example of a geometric sequence. The total area of black triangles at any stage is an example of a geometric series. The value for each term of a series, can be evaluated as the sum of all preceding terms of the geometric sequence.

The general formula for the sum of a geometric sequence is given by \( S_n = \frac{a(1-r^n)}{1-r} \) where

\[ a \] is the first term of the sequence, \( r \) is the common ratio of all terms to the one preceding term, and \( n \) is the number of terms being summed or added.

For the area of the black triangles, \( a = \frac{1}{4}, r = \frac{3}{4} \) and \( n = \) any number of terms to be summed, as this is an expression.

\[
S_n = \frac{\frac{1}{4}(1-(\frac{3}{4})^n)}{1-\frac{3}{4}}
\]

\[
= \frac{\frac{1}{4}(1-(\frac{3}{4})^n)}{\frac{1}{4}}
\]

\[
= 1-(\frac{3}{4})^n
\]

As the question is concerned with the area of the black triangles as \( n \) approaches infinity, the first term must be the area of the first black triangle, which is \( \frac{1}{4} \) units\(^2\). \( a = \frac{1}{4}, r = \frac{3}{4} \).
Therefore, as the process is continued, the total which the infinite series that represents the area of the black triangles approaches is 1 square unit. This can be confirmed by substituting a large \( n \) value into the aforegiven expression for any \( n \) value. This is demonstrated below.

An arbitrary, \( n \) value of 10 is chosen.

Therefore, when \( a = \frac{1}{4} \), and \( r = \frac{3}{4} \), and \( n = 10 \),

\[
S_{10} = 1 - \left( \frac{3}{4} \right)^{10} \approx 0.9437
\]

To show that this process is approaching 1, it should also be demonstrated for another later stage, and the total area should be closer to 1 for this stage. This is worked below for the 12th stage.

Therefore when \( a = \frac{1}{4} \), and \( r = \frac{3}{4} \), and \( n = 12 \),

\[
S_{12} = 1 - \left( \frac{3}{4} \right)^{12} = 0.9683
\]

which is closer to 1 than 0.9437.

Therefore the total area of the black triangles approaches 1, and is equal to 1 at infinity.

5. **By observing the white triangles determine the fraction of the main triangle that is still white at Stage 1, Stage 2, Stage 3 and Stage \( n \).**

By observing the given triangles, it is possible to determine the area of the remaining white triangles at:

*Stage 1, 3/4,*

The remaining white area is 3/4 that of the original white triangle as the area of the black triangle is one quarter that of the original white triangle.

*Stage 2, 9/16,*

The remaining white area is 9/16 that of the original white triangle as the area of the black triangles is 7/16 that of the original white triangle.
Stage 3, 27/64,

The remaining white area is 27/64 that of the original white triangle as the area of the black triangles is 37/64 that of the original white triangle.

The area of white triangles remaining at Stage \( n \), can only be expressed as a geometric progression. To develop an expression for the area at Stage \( n \), the common ratio \( r \), between any two successive areas must be determined. The common ratio is equal to the quotient of the area of white triangles at 3 and 2, and 2 and 1.

The quotient of Stage 3 and 2 is given by

\[
r = \frac{\text{Area of white triangles at stage 3}}{\text{Area of white triangles at stage 2}} = \frac{27/64}{9/16} = \frac{3}{4}
\]

The quotient of Stage 2 and 1

\[
r = \frac{\text{Area of white triangles at stage 2}}{\text{Area of white triangles at stage 1}} = \frac{9/16}{3/4} = \frac{3}{4}
\]

The expression for the area of white triangles at Stage \( n \) can now be developed, using the area of the original triangle, 1 \( \text{unit}^2 \) as the first term \( a \). Therefore, \( a = 1 \) and \( r = 3/4 \). The formula for any term of a geometric sequence is given below.

\[
T_n = ar^{n-1}
\]

However, as \( n \) refers to the stage number and, the area of the first white triangle is used as the first term, because it is represented as stage zero, the formula for any term of a geometric sequence can no longer have \( n - 1 \) as the power of \( r \), so instead should be just \( n \).

Therefore the formula can be expressed as

\[
T_n = ar^n = 1 \times \left(\frac{3}{4}\right)^n = \left(\frac{3}{4}\right)^n
\]
6. Comment on the area of the white triangles as $n \to \infty$.

As $n$ approaches infinity, the area of the white triangles approaches zero. This is because the area of the black triangles approaches one square unit, and the area of the original white triangle was one square unit. This is shown below, at the 10th, 15th and 20th stages.

For the 10th Stage

\[ T_{10} = \left( \frac{3}{4} \right)^{10} \approx 0.0563 \text{ units}^2 \]

For the 15th Stage

\[ T_{15} = \left( \frac{3}{4} \right)^{15} \approx 0.0133 \text{ units}^2 \]

For the 20th Stage

\[ T_{20} = \left( \frac{3}{4} \right)^{20} \approx 0.00317 \text{ units}^2 \]

This shows that as $n$ increases, or approaches infinity, the area of the white triangles, represented by $T_n$, decreases, or approaches zero, where at the 10th stage, the area of the white triangles is 0.0563, and by the 20th stage, it is 0.00317.
Section 2

If the total area added on when the Koch snowflake curve is developed indefinitely, show that it results in a finite area equal to 8/5 of the area of the initial triangle.

The area of the original triangle is assumed to be one units$^2$. In determining areas of successive snowflakes for comparison and calculation, it is necessary to give values of area in terms of the original triangle, and as such, as fractions of the area of the original triangle.

To show that the area of the snowflake, when continued infinitely, results in a finite area equal to 8/5 that of the original triangle, the sum of an infinite sequence can be used.

To begin, the areas of the given snowflakes must be determined, and this information used to evaluate the area of further snowflakes.

The triangles added in stage two are one ninth the area of the original triangle, and three of them are added. Therefore, an area $3/9$ or $1/3$ of the original triangle is added, or $3/9$ units$^2$.

The triangles added in the third stage are one ninth the size of the triangles added in the second stage. Therefore, as the triangles added in the second stage are one ninth the area of the original white triangle, the area of a triangle added in the third stage is one eighty-first the area of the original white triangle, as the product of one ninth and one ninth, is one eighty first, as shown below.

\[
\text{Area of a triangle in stage } 3 = \frac{1}{9} \times \frac{1}{9} = \frac{1}{81}
\]

As two triangles are added to each of the six points of the snowflake in stage two, there are twelve triangles being added in the third stage. Therefore the area of the triangles added in the third stage is $12/81$ the area of the original white triangle, or $12/81$ units$^2$.

In the fourth stage, the triangles added to the sides of the triangles in the previous stage are again one ninth the size that of the previous stage. Therefore, the area of one triangle added in the fourth is the product of $1/9$ and $1/81$.

\[
\text{Area of a triangle in stage } 4 = \frac{1}{81} \times \frac{1}{9} = \frac{1}{729}
\]

Two triangles are added in the fourth stage to each of the twelve triangles added in the previous stage, resulting in 24 triangles being added. Four triangles are also added to each of the six points of the original snowflake, one to the point and base of both sides, which results in another 24 triangles being added. Therefore, in total, there are 48 triangles added in the fourth stage, which results in an increase in area of $48/729$ units$^2$ over the snowflake in the previous stage.
Also, the area of a snowflake at any stage, is the sum of the area of the original triangle, and the area added at each successive stage. Therefore, it can be said, the area of a given snowflake is the sum of the area of the previous snowflake and the area added in the given stage. The total area of the given snowflakes is worked below.

**Stage Two snowflake**

Area of stage 2 snowflake = Area of original triangle + Area added in stage 2

\[
= 1 + \frac{3}{9} = \frac{12}{9}
\]

**Stage Three snowflake**

Area of stage 3 snowflake = Area of stage 2 + Area added in stage 3

\[
= \frac{12}{9} + \frac{12}{81} = \frac{120}{81}
\]

**Stage Four snowflake**

Area of stage 4 snowflake = Area of stage 3 + Area added in stage 4

\[
= \frac{120}{81} + \frac{48}{729} = \frac{1128}{729}
\]

The area added in each stage forms a sequence. It is a geometric sequence as the area added in each stage is a multiple of the last, and the factor is common for every stage. The common factor, or ratio, is given as \( r \). From the table above \( r \) can be determined as the quotient of the area added at stage: 4 and 3; 3 and 2; and 2 and 1.

The quotient of 4 and 3

\[
r = \frac{\text{Area added at stage 4}}{\text{Area added at stage 3}} = \frac{48}{729} = \frac{12}{12} = \frac{81}{9} = \frac{4}{9}
\]
Use of data to synthesise a model in a complex, non-routine situation

Use of problem-solving strategies to interpret and analyse a response to a complex non-routine task in an abstract situation

The quotient of 3 and 2

\[
r = \frac{\text{Area added at stage 3}}{\text{Area added at stage 2}} = 12 \times 81 = \frac{3}{9} = \frac{4}{9}
\]

The quotient of 2 and 1

\[
r = \frac{\text{Area added at stage 2}}{\text{Area added at stage 1}} = 3 \times \frac{9}{1} = \frac{1}{3}
\]

The quotient of the area added in the second stage and the first stage does not correlate with the ratio derived from the quotient of the fourth and third stage, and the third and second stage. Therefore, in calculating the sum to infinity of the series, it is incorrect to use the area added to the original triangle of an area of one unit\(^2\), as the first term of the series. As such, the area added in the second stage must be used as the first term of the sequence, when calculating the area of the snowflake to infinity.

The area of the snowflake at infinity is the sum of the area of the original triangle, and the sum of all area added at infinity. Therefore, as the area of the original triangle is already known, the sum of all areas added at infinity must be determined. It is known that the area added at any stage is a geometric progression.

The common ratio was determined to be \(4/9\). And the first term or \(a\), shall be \(1/3\), the area added to the original triangle in the second stage.

\[
S_\infty \text{ of added area} = \frac{a}{1 - r} = \frac{1}{3} \times \frac{3}{1 - \frac{4}{9}} = \frac{3}{5}
\]

The area of the snowflake when the process is continued indefinitely can now be determined, by adding the area of the original triangle to this value.
Area of snowflake at $\infty = S_\infty$ of added area + Area of original triangle
\[ = \frac{3}{5} + 1 \]
\[ = \frac{8}{5} \]

Therefore it has been proven that the area of the snowflake, when the process is continued indefinitely is equal to the finite area of $\frac{8}{5}$ that of the original triangle, which is $\frac{8}{5}$ units$^2$ when the area of the original triangle is assumed to be 1 unit$^2$.

Show that the Koch snowflake curve has an infinite length, if the process outlined above is continued indefinitely.

The length of each facet should be given in terms of the length of a facet of the original triangle. The length of one facet on the original triangle shall equal one unit.

The question deals with the perimeter of the snowflake. The perimeter is of infinite length as in each snowflake, the sides are forever trisected, a centre third removed, and two more sides of an equilateral triangle added to continue the side with three additional vertices and four sides.

The facets in each successive snowflake decrease in scale by the order of $t$, and therefore represent $1/3$ the length of a facet in the previous snowflake. As one facet is trisected and the triangle added, four facets are formed from one. Therefore from one snowflake to the next, there is four times as many facets, each a third the length of that in the previous snowflake. Which means the perimeter of one snowflake is $4/3$ that of the previous.

The number of facets was counted for each of the given snowflakes. The length of one facet in the Stage Two snowflake, is $1/3$ that of the original triangle. The length of one facet in the Stage Three snowflake is $1/3$ that of the Stage Two triangle, and therefore is $1/3$ of $1/3$, which is $1/9$. The length of one facet in the Stage Four snowflake is again $1/3$ that of the Stage Three snowflake, and so is $1/3$ of $1/9$.

Through discussion above, it is clear that the perimeter of the snowflake is a geometric progression, with a common ratio, or $r$ of $4/3$. Therefore it is possible to determine the perimeter of a snowflake at any stage by deriving a formula for the geometric progression. The working for this is given below.

The formula for a geometric progression is
\[ T_n = ar^{n-1} \]
This will be expressed in terms of perimeter, and so $P_n$. To utilise this expression, $a$ must be determined. It is possible to use the perimeter of the original triangle as the first term, $a$. The perimeter of the first triangle is 3 units, as there are three sides, and it was assumed each side was one unit of length, where all lengths of facets at subsequent stages were given as fractions of this value, and so in terms of the same unit of length. The above formula can now be written for von Koch’s snowflake.
\[ P_n = 3 \left( \frac{4}{3} \right)^{n-1} \]

When the perimeter for snowflakes of increasing stages is calculated, it is clear that the perimeter is continuously increasing. This is demonstrated below for the 10th, 15th and 20th stage.

*Perimeter at the 10th Stage*

\[ P_{10} = 3 \left( \frac{4}{3} \right)^{10-1} \]
\[ = 3 \left( \frac{4}{3} \right)^9 \]
\[ \approx 39.95 \]

*Perimeter at the 15th Stage*

\[ P_{15} = 3 \left( \frac{4}{3} \right)^{15-1} \]
\[ = 3 \left( \frac{4}{3} \right)^{14} \]
\[ \approx 168.37 \]

*Perimeter at the 20th Stage*

\[ P_{20} = 3 \left( \frac{4}{3} \right)^{20-1} \]
\[ = 3 \left( \frac{4}{3} \right)^{19} \]
\[ \approx 709.51 \]

Between the perimeters at the 10th stage, and 15th stage there is a difference of 128.42 units.

Between the perimeters at the 15th stage, and 20th stage there is a difference of 541.14 units.

Between the differences of the perimeter at the three stages there is a difference of 412.72 units. This clearly shows not only an increase in the perimeter of the snowflakes at a constant rate of change, but a variably increasing rate of change. Therefore, if the process is continued indefinitely, the length of the perimeter will continue to increase at a greater rate after every stage, and so will more closely approach infinity at every stage.