Solving differential equations by Integrating factor

V EXAMPLE 1 Solve the differential equation $\dfrac{dy}{dx} + 3x^2y = 6x^2$.

$P(x) = 3x^2 \qquad Q(x) = 6x^2$

$\therefore \quad I(x) = e^{\int 3x^2\,dx} = e^{x^3}$

$\therefore \quad y = \dfrac{1}{e^{x^3}} \int 6x^2 \cdot e^{x^3}$

$y = \dfrac{1}{e^{x^3}}\left(2e^{x^3} + c\right)$

$$\boxed{y = 2 + \frac{c}{e^{x^3}}}$$

V EXAMPLE 2 Find the solution of the initial-value problem

$$x^2y' + xy = 1 \qquad x > 0 \qquad y(1) = 2$$

First rewrite in the standard form for the I.F method.

$$\frac{dy}{dx} + \frac{y}{x} = \frac{1}{x^2}$$

$P(x) = \frac{1}{x} \qquad Q(x) = 1/x^2 \qquad I(x) = e^{\int 1/x} = e^{\ln x} = \underline{\underline{x}}.$

$\therefore \quad y = \dfrac{1}{x}\displaystyle\int x \cdot \frac{1}{x^2} = \frac{1}{x}\left(\ln x + c\right)$

$y = \dfrac{\ln x}{x} + \dfrac{c}{x}$

When $x = 1 \quad y = 2$

$2 = \dfrac{\ln(1)}{1} + \dfrac{c}{1} \qquad \therefore \quad \underline{\underline{c = 2}}$

$y = \dfrac{\ln x}{x} + \dfrac{2}{x}$

EXAMPLE 3 Solve $y' + 2xy = 1$.

$P(x) = 2x \quad Q(x) = 1$

Leave your answer in terms of $\int e^{x^2}\,dx$

$I(x) = e^{\int 2x} = \underline{e^{x^2}}$

$\therefore \quad y = \frac{1}{e^{x^2}} \int e^{x^2}$

Solve:

7. $y' = x - y$

$\frac{dy}{dx} + y = x$

$P(x) = 1 \quad Q(x) = x \quad I(x) = e^{\int 1} = e^{x}$

$\therefore \quad y = \frac{1}{e^x} \int x e^x$

$u = x \quad v' = e^x$

$u' = 1 \quad v = e^x$

$\therefore \quad \int x e^x = x e^x - \int e^x$

$= \underline{x e^x - e^x}$

$y = \frac{1}{e^x}\left(x e^x - e^x + c \right)$

$y = x - 1 + \frac{c}{e^x}$

Solve:

15. $x^2y' + 2xy = \ln x, \quad y(1) = 2$

$$\frac{dy}{dx} + \frac{2y}{x} = \frac{1}{x^2}\ln x$$

$$P(x) = \frac{2}{x} \qquad Q(x) = \frac{1}{x^2}\ln x \qquad I(x) = e^{\int 2/x} = e^{2\ln x} = \underline{\underline{x^2}}$$

$$\therefore\ y = \frac{1}{x^2}\int x^2 \cdot \frac{1}{x^2}\ln x = \frac{1}{x^2}\int \ln x$$

$\int \ln x$ use int by parts. $u = \ln x$, $u' = 1/x$, $v' = 1$, $v = x$. $\int \ln x = x\ln x + -\int 1 = \underline{\underline{x\ln x - x}}$

$$\therefore\ y = \underline{\underline{\frac{x\ln x - x}{x^2}}}$$

$$2 = \frac{-1+c}{1} \quad \therefore c =$$

$$y = \frac{\ln x}{x} - \frac{1}{x} + \frac{3}{x^2}$$

Example 46

Solve the initial value problem $\cos x\,\dfrac{dy}{dx} = y\sin x + \sin(2x), \quad y(0) = 1.$

$$\frac{dy}{dx} + -y\frac{\sin x}{\cos x} = \frac{\sin(2x)}{\cos(x)} = \frac{2\sin x\cancel{\cos x}}{\cancel{\cos x}}$$

$$\frac{dy}{dx} - y\tan x = 2\sin(x).$$

$$P(x) = -\tan(x) \qquad Q(x) = 2\sin x$$

$$I(x) = e^{\int -\tan(x)} = e^{+\ln(\cos x)} = \underline{\underline{\cos x}}$$

$$\therefore\ y = \frac{1}{\cos x}\int \cos x \cdot 2\sin x = \frac{1}{\cos x}\cdot \int \sin(2x)$$

$$y = \frac{1}{\cos x}\left((-)\frac{\cos 2x}{2} + c\right)$$

$$1 = 1\left(-\frac{1}{2} + c\right) \quad \therefore c = 3/2$$

and $y = \dfrac{-\cos 2x}{2\cos x} + \dfrac{3}{2\cos x}$