Homogenous differential equations

a Use the substitution $y = vx$, where v is a function of x, to solve:

$$\frac{dy}{dx} = \frac{x+2y}{x}$$

b Find the particular solution if $y = \frac{3}{2}$ when $x = 3$.

$y = vx$

$\frac{dy}{dx} = v + \frac{dv}{dx}x = \frac{x+2y}{x} = 1 + 2\left(\frac{y}{x}\right) = 1 + 2v$

$v + \frac{dv}{dx}x = 1 + 2v \quad \therefore \quad \frac{dv}{dx}x = 1 + v \quad \therefore \int \frac{dv}{1+v} = \int \frac{dx}{x}$

$\ln(1+v) = \ln x + c$

$\ln(1 + y/x) = \ln x + c$

$1 + y/x = Ax$

$y = Ax^2 - x$

$\frac{3}{2} = A(3^2) - 3$

$A = 1/2$

$\boxed{y = \frac{1}{2}x^2 - x}$

6 Solve the homogeneous differential equations below using the substitution $y = vx$, where v is a function of x.

a $$\frac{dy}{dx} = \frac{x-y}{x}$$

$\frac{dv}{dx}x + v = 1 - v$

$\int \frac{dv}{1-2v} = \int \frac{dx}{x} \quad \therefore -\frac{1}{2}\ln(1-2v) = \ln(x) + c$

$\left(1 - 2\left(\frac{y}{x}\right)\right)^{-1/2} = Ax$ etc.

$1 - \frac{2y}{x} = Ax^{-2}$

$x^2 - 2xy = A$

b $$\frac{dy}{dx} = \frac{x+y}{x-y}$$

$\frac{dv}{dx}x + v = \frac{x + vx}{x - vx} = \frac{x(1+v)}{x(1-v)}$

$\frac{dv}{dx}x + v = \frac{1+v}{1-v}$

$\frac{dv}{dx}x = \frac{1+v}{1-v} - v$

$\frac{dv}{dx}x = \frac{1+v}{1-v} - \frac{v(1-v)}{(1-v)} = \frac{1+v^2}{1-v}$

$dv \cdot \frac{1-v}{1+v^2} = dx\,\frac{1}{x}$

$\left(\frac{1}{1+v^2} - \frac{v}{1+v^2}\right)dv = \frac{1}{x}\,dx$

$\arctan|v| - \frac{1}{2}\ln(1+v^2) = \ln x + c$

$\arctan\left(\frac{y}{x}\right) - \frac{1}{2}\ln\left(1+\left(\frac{y}{x}\right)^2\right) = \ln x + c$

or

c $$\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$$

$$\frac{dv}{dx}x + v = \frac{y}{2x} - \frac{x}{2y} = \frac{1}{2}v - \frac{1}{2}\frac{1}{v}$$

$$\frac{dv}{dx}x = -\frac{1}{2}v - \frac{1}{2}\frac{1}{v} = \frac{-1}{2}\left(\frac{v^2+1}{v} \right)$$

$$dv \frac{v}{v^2+1} = \left(\frac{-1}{2}\right)\frac{1}{x}\,dx \qquad \therefore \frac{1}{2}\ln(v^2+1) = \frac{-1}{2}(\ln x + c)$$

$$\ln\left(\left(\tfrac{y}{x}\right)^2+1\right) = -\ln x + c$$

$$\left(\frac{y}{x}\right)^2 + 1 = Ax^{-1}$$

$$y^2 + x^2 = Ax.$$

Solve $x\dfrac{dy}{dx} = y + e^{\frac{y}{x}}$

$$\frac{dy}{dx} = \frac{y}{x} + \frac{1}{x}e^{y/x}$$

$$\frac{dv}{dx}x + v = v + \frac{1}{x}\cdot e^{v}$$

$$\frac{dv}{dx}\cdot x = \frac{1}{x}e^{v}$$

$$\int \frac{dv}{e^v} = \int \frac{dx}{x^2}$$

$$-e^{-v} = -x^{-1} + c$$

$$e^{-y/x} = x^{-1} + c$$

$$-y/x = \ln(x^{-1} + c)$$

$$y = x\ln(x^{-1} + c)^{-1}$$